Population models

To model a population, let $P(t)$ be its size at time t.

$\beta(t)$, $\delta(t)$: birth and death rate [# of births/deaths (per unit of population per unit of time) at time t]
$\Delta P = \beta(t)P(t)\Delta t - \delta(t)P(t)\Delta t$
$\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t))P$

Example 31. Some assumptions and corresponding models. [We'll come back here next class!]

- **(basic)** If $\beta(t)$ and $\delta(t)$ are constant, we get the exponential model $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$. $P(t) = Ce^{kt}$.
- **(limited supply)** $\delta(t)$ constant, $\beta(t) = \beta_0 - \beta_1 P$
  $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta_0 - \beta_1 P - \delta)P = aP - bP^2 = kP(1 - P/M)$. This is the logistic equation from Lecture 2.
- **(rare species)** $\delta(t)$ constant, $\beta(t)$ proportional to $P(t)$
  $\frac{\mathrm{d}P}{\mathrm{d}t} = (\gamma P - \delta)P$. The logistic equation, again.
- **(rare species with very long life)** $\delta(t) = 0$, $\beta(t)$ proportional to $P(t)$
  $\frac{\mathrm{d}P}{\mathrm{d}t} = kP^2$. Solutions are $P(t) = \frac{1}{C - kt}$ where $P(0) = 1/C$.
  This explodes when $t \to C/k$. (But by then the species is not exactly rare anymore...)
- **(harvesting)** Each unit of time, h population units are harvested.
  $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t))P - h$
  For instance, $\frac{\mathrm{d}P}{\mathrm{d}t} = kP - h$ has $P(t) = Ce^{kt} + h/k$.
- **(spread of incurable virus)** Let $P(t)$ count the number of infected population units among total of M.
  $\delta(t) = 0$, $\beta(t)$ proportional to $M - P$
  $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(M - P)$. Once again, the logistic equation. ◇

Example 32. Solve the logistic equation $P' = kP(1 - P/M)$. Separable!

Solution. $\frac{-M}{P(P - M)}\mathrm{d}P = \left(\frac{1}{P} - \frac{1}{P - M}\right)\mathrm{d}P = k\mathrm{d}t$. We get $\ln|P| - \ln|P - M| = \ln\left|\frac{P}{P - M}\right| = kt + C$.
Hence, $\frac{P}{P - M} = De^{kt}$ with $D = \pm e^C$. Thus $P(t) = \frac{MDe^{kt}}{De^{kt} - 1}$. [cf. Example 7.] ◇

Linear higher-order differential equations

A linear DE is of the form $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = f(x)$.

- Let I be an interval on which $p_j(x)$ and $f(x)$ are continuous. If $a \in I$ then a solution to the IVP with $y(a) = b_0$, $y'(a) = b_1$, ..., $y^{(n-1)}(a) = b_{n-1}$ always exists (actually, on all of I!) and is unique.

If $f(x) = 0$, then this is called a homogeneous linear DE. In that case:

- If y_1 and y_2 are solutions, then the superposition $Ay_1 + By_2$ is a solution.
- **(general solution)** There are n solutions $y_1, y_2, \ldots, y_n$, such that every solution is of the form $C_1 y_1 + \ldots + C_n y_n$. [These n solutions necessarily are, what we will call, independent.]

Example 33. Suppose that y_1 and y_2 solve $y'' + p_1(x)y' + p_0(x)y = 0$.

$(y_1 + y_2)'' + p_1(x)(y_1 + y_2)' + p_0(x)(y_1 + y_2) = \{y_1'' + p_1(x)y_1' + p_0(x)y_1\} + \{y_2'' + p_1(x)y_2' + p_0(x)y_2\} = 0 + 0$
In other words, $y_1 + y_2$ is another solution of the DE. ◇

Example 34. $x^2y'' + 2xy' - 6y = 0$ has solutions $y_1 = x^2$, $y_2 = x^{-3}$.
Solve the IVP with $y(2) = 10$, $y'(2) = 15$.

Solution. The general solution is $y(x) = Ax^2 + Bx^{-3}$. $y'(x) = 2Ax - 3Bx^{-4}$.
$y(2) = 4A + B/8 = 10$, $y'(2) = 4A - 3/16B = 15$ has solutions $A = 3$, $B = -16$. So $y(x) = 3x^2 - 16/x^3$. ◇