Review. complex numbers ◇

Example 51. Here is another way, to look at Euler's identity $e^{ix} = \cos(x) + i\sin(x)$.

For this identity to make sense, one needs to somehow characterize the exponential function on the left-hand side. Last time, we observed that both sides are the unique solution to the IVP $y'' + y = 0$, $y(0) = 1$, $y'(0) = i$. This time, we use the Taylor expansion $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Note that i^n, for $n = 0, 1, 2, \ldots$, is $1, i, -1, -i, 1, i, \ldots$. Hence, splitting the Taylor sum into even $(n = 2m)$ and odd $(n = 2m + 1)$ terms, we get

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = \cos(x) + i\sin(x).$$ ◇

Example 52. Euler's identity makes deriving many trig formulas easy!

For instance, $\cos(x+y) = \mathrm{Re}(e^{i(x+y)}) = \mathrm{Re}(e^{ix}e^{iy}) = \mathrm{Re}((\cos(x) + i\sin(x))(\cos(y) + i\sin(y)))$
$= \cos(x)\cos(y) - \sin(x)\sin(y)$. ◇

Example 53. Express the general solution of $y''' - y'' + 4y' - 4y = 0$ using only real functions.

Solution. Characteristic polynomial $r^3 - r^2 + 4r - 4$. We spot the root 1. To find the other roots, we do polynomial division to get $(r^3 - r^2 + 4r - 4)/(r - 1) = r^2 + 4$. Hence, the characteristic polynomial has roots $1, \pm 2i$.

We therefore have the following solutions $y_1 = \cos(2x)$, $y_2 = \sin(2x)$, $y_3 = e^x$. The general solution is $A\cos(2x) + B\sin(2x) + Ce^x$.

Our next goal is to think deeper about the final step, which allowed us to go from having three solutions to the general solution. ◇

Independence of solutions

Given a homogeneous linear DE, we have learned that there *exist* n solutions $y_1, y_2, \ldots, y_n$, such that the general solution is $C_1 y_1 + \ldots + C_n y_n$. But if we find n solutions, how can we tell whether they give the general solution?

Example 54. Here are three other solutions of the previous example: $u_1 = \cos(2x)$, $u_2 = \sin(2x)$, $u_3 = \cos(2x + 1)$. However, $c_1 u_1 + c_2 u_2 + c_3 u_3$ is not the general solution to the DE. Why?

Using the trig identity from Example 52, we see that $\cos(1)u_1 - \sin(1)u_2 - u_3 = 0$. This is a (linear) dependence relation between the solution functions and allows us to express one of them in terms of the other two. In other words, we really have only two solutions (the "really" means that we have only two independent solutions, a notion defined below) and are still missing a third one to get the general solution. ◇

Definition 55. n functions $f_1, \ldots, f_n$ are (linearly) dependent if there are coefficients $c_1, \ldots, c_n$, not all zero, such that $c_1 f_1 + \ldots + c_n f_n = 0$. Otherwise, they are called (linearly) independent.

Theorem 56. Suppose that $y_1, y_2, \ldots, y_n$ are solutions to a homogeneous linear DE of order n. Then the general solution is $C_1 y_1 + \ldots + C_n y_n$ if and only if $y_1, y_2, \ldots, y_n$ are independent.

Example 57. Are the functions $u_1 = 3x^2\sin^2(x)$, $u_2 = 5x^2\cos^2(x)$ and $u_3 = x^2$ linearly independent?

Solution. Using $\cos^2(x) + \sin^2(x) = 1$, we find that $\frac{1}{3}u_1 + \frac{1}{5}u_2 - u_3 = 0$. In other words, the functions are linearly dependent. ◇

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