Review. linear independence ◇

Fix some $a \in I$. Note that $y(x) = C_1 y_1(x) + \ldots + C_n y_n(x)$ is the general solution of a HLDE[8] of order n if and only if we can solve for all initial values $y(a) = b_0$, $y'(a) = b_1$, …, $y^{(n-1)}(a) = b_{n-1}$.

Writing out these (linear) equations and expressing them in matrix form, we see that they are equivalent to finding $(C_1, C_2 \ldots, C_n)$ such that

$$\begin{pmatrix} y_1(a) & y_2(a) & \cdots & y_n(a) \\ y_1'(a) & y_2'(a) & \cdots & y_n'(a) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(a) & y_2^{(n-1)}(a) & \cdots & y_n^{(n-1)}(a) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix}.$$

Linear Algebra[9] tells us that this system of linear equations can be solved for all values of the b_j if and only if the determinant of the matrix on the LHS is not zero. This determinant is the Wronskian $W(a)$ of $y_1, \ldots, y_n$.

Definition 58. The Wronskian of the n functions $f_1, \ldots, f_n$ is the $n \times n$ determinant

$$W(x) := \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

Note that, for linearly dependent functions, $W(x) = 0$ for all x. Why?!

Theorem 59. Solutions $y_1, \ldots, y_n$ of a homogeneous linear DE of order n are linearly independent if and only if $W(x) \neq 0$ for some $x \in I$. [in which case $W(x) \neq 0$ for all $x \in I$]

Example 60. $y'' + 4y' + 4y = 0$ has solutions $y_1 = e^{-2x}$, $y_2 = x e^{-2x}$.

The Wronskian is $\begin{vmatrix} e^{-2x} & x e^{-2x} \\ -2e^{-2x} & (1-2x) e^{-2x} \end{vmatrix} = e^{-2x}(1-2x)e^{-2x} - (-2e^{-2x}) x e^{-2x} = e^{-4x}[1 - 2x + 2x] = e^{-4x} \neq 0.$

Hence, y_1, y_2 are independent and the general solution is $y(x) = A y_1(x) + B y_2(x)$. ◇

Example 61. $y''' = 0$ has solutions $y_1 = 3$, $y_2 = 1 - 2x^2$, $y_3 = 5x^2$. Are these independent?

Solution. No, because $y_1 - 3y_2 - \frac{6}{5} y_3 = 0$.

Solution. No, because $W(0) = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -4 & 10 \end{vmatrix} = 0.$ [evaluating the Wronskian at 0 makes our computation easy!]

What about the solutions $y_1 = 3$, $y_2 = 1 - 2x^2$, $y_3 = 5x$. Are they independent?

Solution. Yes, because $W(0) = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 0 & 5 \\ 0 & -4 & 0 \end{vmatrix} = 60.$ ◇

Remark. JFF[10]. The Riemann zeta function is defined by the sum $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, which converges if Re $s > 1$. For other complex values of s, there is a unique way to "analytically continue" this function. It is then "easy" to see that $\zeta(-2) = 0$, $\zeta(-4) = 0$, …. The Riemann hypothesis claims that all other zeroes of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$. A proof of this conjecture (checked for the first 10,000,000,000 zeroes) is worth[11] \$1,000,000. ◇

8. Writing the DE as $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = 0$, we need the coefficients $p_j(x)$ to be at least continuous on the interval I.

9. Don't worry if you are not familiar with this, as we will go over basics of Linear Algebra when we really need it. However, it may be a good idea to start reading up on matrices and vectors because we will be brief.

10. Just for fun.

11. `http://www.claymath.org/millenium-problems/riemann-hypothesis`

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