Review. solving nonhomogeneous linear DEs with constant coefficients ◇

Example 70. Find a particular solution of $y'' + 4y' + 4y = 7e^{-2x}$.

Solution. Again, $L = D^2 + 4D + 4 = (D+2)^2$.
"Old" roots $-2, -2$. "New" roots -2. Hence, there has to be a particular solution of the form $y_p = Cx^2e^{-2x}$. To find the value of C, we need to plug into the DE.

$y_p' = C(-2x^2 + 2x)e^{-2x}$

$y_p'' = C(4x^2 - 8x + 2)e^{-2x}$

$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} \overset{!}{=} 7e^{-2x}$. $C = 7/2$.

Hence, $y_p = \frac{7}{2}x^2e^{-2x}$. (Last time, we didn't finish the computation.) ◇

Example 71. Find a particular solution of $y'' + 4y' + 4y = x\cos(x)$.

Solution. "Old" roots $-2, -2$. "New" roots $\pm i, \pm i$. Hence, there has to be a particular solution of the form $y_p = (C_1 + C_2x)\cos(x) + (C_3 + C_4x)\sin(x)$. To find the value of the C_j's, we need to plug into the DE.

$y_p' = (C_2 + C_3 + C_4x)\cos(x) + (C_4 - C_1 - C_2x)\sin(x)$

$y_p'' = (2C_4 - C_1 - C_2x)\cos(x) + (-2C_2 - C_3 - C_4x)\sin(x)$

$y_p'' + 4y_p' + 4y_p = (3C_1 + 4C_2 + 4C_3 + 2C_4 + (3C_2 + 4C_4)x)\cos(x)$
$+ (-4C_1 - 2C_2 + 3C_3 + 4C_4 + (-4C_2 + 3C_4)x)\sin(x) \overset{!}{=} x\cos(x)$.

Equating the coefficients of $\cos(x)$, $x\cos(x)$, $\sin(x)$, $x\sin(x)$, we get the equations $3C_1 + 4C_2 + 4C_3 + 2C_4 = 0$, $3C_2 + 4C_4 = 1$, $-4C_1 - 2C_2 + 3C_3 + 4C_4 = 0$, $-4C_2 + 3C_4 = 0$.

Solving, we find $C_1 = -\frac{4}{125}$, $C_2 = \frac{3}{25}$, $C_3 = -\frac{22}{125}$, $C_4 = \frac{4}{25}$. [Make sure you know how to do this tedious step.]

Hence, $y_p = \left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)$. ◇

Example 72. Find a particular solution of $y'' + 4y' + 4y = 5e^{-2x} - 3x\cos(x)$.

Solution. Instead of starting all over, recall that we already found $y_\triangle$ in Example 70 such that $Ly_\triangle = 7e^{-2x}$. Also, from Example 71 we have $y_\diamond$ such that $Ly_\diamond = x\cos(x)$.

By linearity, it follows that $L\left(\frac{5}{7}y_\triangle - 3y_\diamond\right) = \frac{5}{7}Ly_\triangle - 3Ly_\diamond = 5e^{-2x} - 3x\cos(x)$.

Hence, $y_p = \frac{5}{7}y_\triangle - 3y_\diamond = \frac{5}{2}x^2e^{-2x} - 3\left[\left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)\right]$. ◇

Example 73. Find a particular solution of $y'' + 4y' + 4y = 4e^{3x}\sin(2x) - x\sin(x)$.

Solution. "Old" roots $-2, -2$. "New" roots $3 \pm 2i, \pm i, \pm i$.

Hence, there has to be a particular solution of the form

$y_p = C_1e^{3x}\cos(2x) + C_2e^{3x}\sin(2x) + (C_3 + C_4x)\cos(x) + (C_5 + C_6x)\sin(x)$.

To find the values of $C_1, \ldots, C_6$, we plug into the DE. But this final step is so boring that we stop here.

Computers (currently?) cannot afford to be as selective; mine obediently calculated:

$y_p = -\frac{4}{841}e^{3x}(20\cos(2x) - 21\sin(2x)) + \frac{1}{125}((-22 + 20x)\cos(x) + (4 - 15x)\sin(x))$ ◇

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