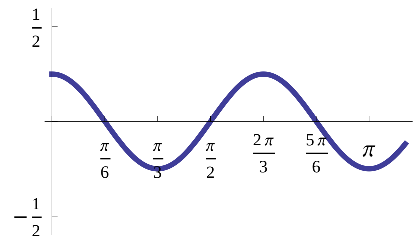
Review. Let θ be angular displacement of a pendulum on a string of length L. Then its motion is described by $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$. $\sin\theta \approx \theta$ for small θ so, approximately[13], we get $\theta'' + \frac{g}{L}\theta = 0$. (This time, we used Newton's second law and considered the tangential component of the gravitational force to derive the equation of motion. $F = -\sin\theta \cdot mg$ and $F = ma = mL\theta''$.) ◇

Example 78. The motion of a mass attached to a spring $mx'' + kx = 0$.

Here, x is displacement from equilibrium, k spring constant (for instance, N/m), m mass. Hooke's law: $F = -kx$. Newton: $F = ma = mx''$. ◇

Example 79. Solve the IVP $\theta'' + 9\theta = 0$, $\theta(0) = 1/4$ (about 14.3°), $\theta'(0) = 0$.

Solution. The roots of the characteristic polynomial are $\pm 3i$.
Hence, $\theta(t) = A\cos(3t) + B\sin(3t)$. $\theta(0) = A = 1/4$. $\theta'(0) = 3B = 0$.
Therefore, the solution is $\theta(t) = 1/4\cos(3t)$
Amplitude: 1/4
Period: $T = \frac{2\pi}{3}$
(Circular) frequency: 3



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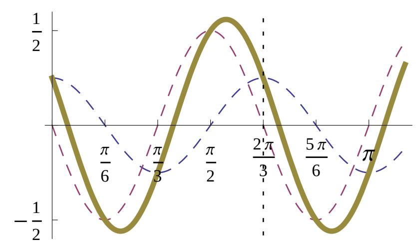
Example 80. Solve $\theta'' + 9\theta = 0$, $\theta(0) = 1/4$, $\theta'(0) = -\frac{3}{2}$ ("initial kick"). What is the amplitude?

Solution. This time, $\theta(0) = A = 1/4$. $\theta'(0) = 3B = -3/2$.
Hence, $\theta(t) = \frac{1}{4}\cos(3t) - \frac{1}{2}\sin(3t) = \frac{\sqrt{5}}{4}\cos(3t - \alpha)$.
The last equality follows because $\left(\frac{1}{4}, -\frac{1}{2}\right) = r(\cos\alpha, \sin\alpha)$ with $r = \frac{\sqrt{5}}{4}$ and $\alpha = \tan^{-1}(-2) + 2\pi \approx 5.176$. (see next Example and Review)
The amplitude is $\frac{\sqrt{5}}{4} \approx 0.559$.
Phase angle: α (or time lag $\alpha/3$)



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Example 81. $A\cos(\omega t) + B\sin(\omega t) = r\cos(\omega t - \alpha)$ with (r, α) the polar coordinates for (A, B)

If you like trig identities: $A\cos(\omega t) + B\sin(\omega t) = r(\cos(\alpha)\cos(\omega t) + \sin(\alpha)\sin(\omega t)) = r\cos(\omega t - \alpha)$.
If you like DEs: both sides solve $x'' + \omega x = 0$. The LHS has initial values $y(0) = A$ and $y'(0) = \omega B$, the RHS has $y(0) = r\cos(\alpha)$ and $y'(0) = r\omega\sin(\alpha)$. Hence, the two are equal if $A = r\cos(\alpha)$ and $B = r\sin(\alpha)$. ◇

Review 82. How to calculate the polar coordinates (r, α) for (A, B)?

We need to find $r \geqslant 0$ and $\alpha \in [0, 2\pi)$ such that $(A, B) = r(\cos\alpha, \sin\alpha)$. Hence, $r = \sqrt{A^2 + B^2}$ and α is determined by $\cos(\alpha) = \frac{A}{r}$ and $\sin(\alpha) = \frac{B}{r}$. In particular, $\tan(\alpha) = \frac{B}{A}$ and, if careful, we can compute α using $\tan^{-1}$ as

$$\alpha = \tan^{-1}\left(\frac{B}{A}\right) + \begin{cases} 0, & \text{if } (A, B) \text{ in first quadrant,} \\ 2\pi, & \text{if } (A, B) \text{ in fourth quadrant,} \\ \pi, & \text{otherwise.} \end{cases}$$

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13. At least for short times and small angles. For instance, for $\theta = 15°$ the error $\theta - \sin\theta$ is about 1%.

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