The qualitative effects of damping

Let us consider $x'' + dx' + cx = 0$ with $c > 0$ and $d \geqslant 0$. The term dx' models damping (e.g. friction, air resistance) proportional to the velocity x'.

The characteristic equation $r^2 + dr + c = 0$ has roots $\frac{-d \pm \sqrt{d^2 - 4c}}{2}$. The nature of the solutions depends on whether the discriminant $\Delta = d^2 - 4c$ is positive, negative, or zero.

Undamped. $d = 0$. In that case, $\Delta < 0$. Two complex roots $\pm i\omega$ with $\omega = \sqrt{c}$.

Solutions: $c_1 \cos(\omega t) + c_2 \sin(\omega t) = r\cos(\omega t - \alpha)$ where $(c_1, c_2) = r(\cos\alpha, \sin\alpha)$
Oscillations with frequency $\omega = \sqrt{c}$, period $\frac{2\pi}{\sqrt{c}}$, time lag $\frac{\alpha}{\sqrt{c}}$

Underdamped. $d > 0$, $\Delta < 0$. Two complex roots $-\rho \pm i\omega$ with $-\rho = -d/2 < 0$.

Solutions: $e^{-\rho t}[c_1\cos(\omega t) + c_2\sin(\omega t)] = e^{-\rho t}[r\cos(\omega t - \alpha)]$ ($\to 0$ as $t \to \infty$)
Oscillations with amplitude going to zero

Critically damped. $d > 0$, $\Delta = 0$. One (double) real root $-\rho < 0$.

Solutions: $(c_1 + c_2 t)\, e^{-\rho_1 t}$ ($\to 0$ as $t \to \infty$)
No oscillations (at most one crossing of t-axis; why?!)

Overdamped. $d > 0$, $\Delta > 0$. Two real roots $-\rho_1, -\rho_2 < 0$. [negative because $c, d > 0$]

Solutions: $c_1 e^{-\rho_1 t} + c_2 e^{-\rho_2 t}$ ($\to 0$ as $t \to \infty$)
No oscillations (at most one crossing of t-axis)

Adding external forces and the phenomenon of resonance

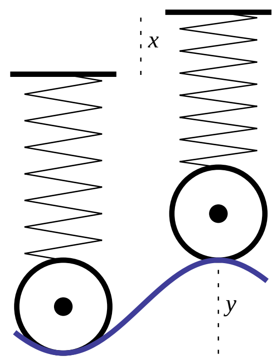
Example 83. A car is going at constant speed v on a washboard road surface ("bumpy road") with height profile $y(s) = a\cos\left(\frac{2\pi s}{L}\right)$. Assume that the car oscillates vertically as if on a spring (no dashpot). Describe the resulting oscillations.

Solution. With x as in the sketch, the spring is stretched by $x - y$. Hence, by Hooke's and Newton's laws, its motion is described by $mx'' = -k(x - y)$.

At constant speed, $s = vt$ and we obtain the DE $mx'' + kx = ky = ka\cos\left(\frac{2\pi v t}{L}\right)$, which is inhomogeneous linear with constant coefficients. Let's solve it.

"Old" roots $\pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$. $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency (the frequency at which the system would oscillate in the absence of external forces).

"New" roots $i\frac{2\pi v}{L} = \pm i\omega$. $\omega = \frac{2\pi v}{L}$ is the external frequency.



The first "car" is assumed to be in equilibrium.

Case 1: $\boldsymbol{\omega \neq \omega_0}$. Then a particular solution is $x_p = b_1\cos(\omega t) + b_2\sin(\omega t) = A\cos(\omega t - \alpha)$ for unique values of b_1, b_2 (which we do not compute here). The general solution is of the form $x = x_p + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$.

Case 2: $\boldsymbol{\omega = \omega_0}$. Then a particular solution is $x_p = t[b_1\cos(\omega t) + b_2\sin(\omega t)] = At\cos(\omega t - \alpha)$ for unique values of b_1, b_2 (which we do not compute). Note that the amplitude in x_p increases without bound; the same is true for the general solution $x = x_p + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$. This phenomenon is called resonance; it occurs if an external frequency matches a natural frequency.

$\diamond$

Note that resonance (or anything close to it) is very important for practical purposes because large amplitudes can be very destructive: singing to shatter glass, earth quake waves and buildings, marching soldiers on bridges, ...

Example 84. Consider $x'' + 9x = 10\cos(2vt)$. For what value of v does resonance occur?

Solution. The natural frequency is 3. The external frequency is $2v$. Hence, resonance occurs when $v = \frac{3}{2}$. $\diamond$

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