Review. properties of determinants ◇

The determinant of any matrix can be computed by picking a row i and calculating $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{[i,j]})$, where $A_{[i,j]}$ is obtained from A by deleting the ith row and jth column.

The determinant satisfies $\det(A^T) = \det(A)$ (as a consequence, we can adjust the above formula to expand along columns instead of along a row) and $\det(AB) = \det(A)\det(B)$.

Example 101. $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ ◇

Vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are (linearly) independent if $c_1\boldsymbol{x}_1 + \ldots + c_n\boldsymbol{x}_n = 0$ only for $c_1 = c_2 = \ldots = c_n = 0$.

When checking independence of n many $n \times 1$ column vectors, we can use determinants! They are independent if and only if $\det(\begin{array}{cccc} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \ldots & \boldsymbol{x}_n \end{array}) \neq 0$. [Do you see why?!]

Example 102. Are the vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ independent?

Solution. $\det\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = -3 \neq 0$. Hence the vectors are independent. ◇

Example 103. Are the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ independent?

Solution. $\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 1 & 4 & 0 \end{pmatrix} = 2\det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - (-1)\det\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} = -2 + 2 = 0$. Hence the vectors are dependent.

Solution. $4\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0$. So, by definition, they are dependent. ◇

Review. We took another look at Theorem 59 from Lecture 13 to convince ourselves that it follows from what we just learned about determinants. ◇

Example 104. $x'' - x' - 2x = 0$ has solutions $x_1 = e^{2t}$, $x_2 = e^{-t}$.

Since $W(t) = \det\begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix}$ and so $W(0) = \det\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = -3 \neq 0$, our two solutions are indeed independent. As a consequence, the general solution is $c_1x_1 + c_2x_2$. ◇

Example 105. Introducing $y = x'$, the previous DE is equivalent to the first-order system $x' = y$, $y' = 2x + y$. Our known solutions translate into $x_1 = e^{2t}$, $y_1 = 2e^{2t}$ and $x_2 = e^{-t}$, $y_2 = -e^{-t}$.

Writing $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, this system is $\boldsymbol{x}' = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\boldsymbol{x}$ with solutions $\boldsymbol{x}_1 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$, $\boldsymbol{x}_2 = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$.

For fixed t, these two vectors are independent if and only if $\det(\begin{array}{cc} \boldsymbol{x}_1 & \boldsymbol{x}_2 \end{array}) = \det\begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix} \neq 0$. Note that this is precisely the Wronskian of the previous example. ◇

As in this last example, if $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are solutions to $\boldsymbol{x}' = A(t)\boldsymbol{x}$ (a homogeneous linear first-order system of DEs), then their Wronskian is the determinant $W(t) = \det(\begin{array}{cccc} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \ldots & \boldsymbol{x}_n \end{array})$. Next time, we will see the expected properties it again has.

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