Review. solutions to the sytem discussed last time ◇

In order to solve $\boldsymbol{x}' = A\boldsymbol{x}$, we look for solutions $\boldsymbol{x}(t) = \boldsymbol{v}e^{\lambda t}$.

Plugging into the DE, we get $\boldsymbol{x}' = \boldsymbol{v}\lambda e^{\lambda t} \overset{!}{=} Ae^{\lambda t}\boldsymbol{v}$. Cancelling the exponentials, we see that we have a solution if and only if $A\boldsymbol{v} = \lambda\boldsymbol{v}$.

Definition 110. If $A\boldsymbol{v} = \lambda\boldsymbol{v}$, for $\boldsymbol{v} \neq 0$, then $\boldsymbol{v}$ is an eigenvector of A with eigenvalue λ.

In order to find these, note that $A\boldsymbol{v} = \lambda\boldsymbol{v} = \lambda I\boldsymbol{v}$ is equivalent to $(A - \lambda I)\boldsymbol{v} = 0$. By the properties of determinants, this is only possible if $\det(A - \lambda I) = 0$. This determinant is a polynomial in λ, the characteristic polynomial of A. Its roots are the eigenvalues λ.
For a specific eigenvalue λ, we then solve $(A - \lambda I)\boldsymbol{v} = 0$ to find the eigenvector(s) $\boldsymbol{v}$.

Example 111. Find the general solution of $\boldsymbol{x}' = \begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix}\boldsymbol{x}$.

Solution. The characteristic polynomial is

$$\det\left[\begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \det\begin{pmatrix} 4-\lambda & 2 \\ -3 & -1-\lambda \end{pmatrix} = (4-\lambda)(-1-\lambda) + 6 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2).$$

This means that the eigenvalues are $\lambda = 1$ and $\lambda = 2$.

$\boldsymbol{\lambda = 1}$**.** To find $\boldsymbol{v}$, we have to solve $\begin{pmatrix} 3 & 2 \\ -3 & -2 \end{pmatrix}\boldsymbol{v} = 0$. We find $\boldsymbol{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, or any multiple thereof.

If that was too fast for you, note that we need to solve $3v_1 + 2v_2 = 0$, $-3v_1 - 2v_2 = 0$. The second equation is worthless; it is just the first one times -1. Hence, we are free to set, for instance, $v_1 = c$. The equations then give $v_2 = -\frac{3}{2}c$. The most general solution therefore is $\boldsymbol{v} = (c, -\frac{3}{2}c)^T$. Our eigenvector above is the choice $c = 2$. [Why do we not care about which multiple of the eigenvector to pick?!]

$\boldsymbol{\lambda = 2}$**.** Now, we have to solve $\begin{pmatrix} 2 & 2 \\ -3 & -3 \end{pmatrix}\boldsymbol{v} = 0$. We find $\boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Consequently, we have solutions $\boldsymbol{x}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}e^t$ and $\boldsymbol{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t}$.

Let us check that these are independent using the Wronskian.

$$W(t) = \det\begin{pmatrix} 2e^t & e^{2t} \\ -3e^t & -e^{2t} \end{pmatrix}, \quad W(0) = \det\begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix} = 1 \neq 0.$$

This certifies that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are independent.

Therefore, the general solution is $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = c_1\begin{pmatrix} 2 \\ -3 \end{pmatrix}e^t + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} = \begin{pmatrix} 2c_1e^t + c_2e^{2t} \\ -3c_1e^t - c_2e^{2t} \end{pmatrix}$. ◇

Remark 112. (JustForFun)

Recall that, abstractly, vectors are anything that can be added and scaled. In that abstract sense, matrices are functions (better, operators), which take vectors as input and return vectors as output (a matrix A takes the vector $\boldsymbol{x}$ and returns the vector $A\boldsymbol{x}$). These operators are linear, meaning that, for instance, $A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y}$. Now, think of (differentiable) functions on the real line. They can be added and scaled, and so form a vector space. There is a very interesting and basic linear operator: the derivative D, which takes a function f and returns $Df = f'$.

What are the eigenfunctions[16] f and eigenvalues λ of D? That is, what are the solutions to $Df = \lambda f$? For any λ, there is a solution: the exponential $f = e^{\lambda x}$ (or multiples thereof).

In other words, any λ is an eigenvalue of D and $e^{\lambda x}$ is a corresponding eigenfunction. In short, the *exponentials are important because they are the eigenfunctions of the derivative*! ◇

16. That is just a more politically correct name for eigenvector in this context.

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