**Review.** eigenvectors, eigenvalues and corresponding solutions to systems ♢

**Example 113.** Find the general solution of $\boldsymbol{x}' = \begin{pmatrix} -5 & 3 & 3 \\ 0 & -1 & 2 \\ -6 & 5 & 2 \end{pmatrix} \boldsymbol{x}$.

**Solution.** The characteristic polynomial is

$$\det \begin{pmatrix} -5-\lambda & 3 & 3 \\ 0 & -1-\lambda & 2 \\ -6 & 5 & 2-\lambda \end{pmatrix} = (-1-\lambda)\det\begin{pmatrix} -5-\lambda & 3 \\ -6 & 2-\lambda \end{pmatrix} - 2\det\begin{pmatrix} -5-\lambda & 3 \\ -6 & 5 \end{pmatrix} = \ldots = -\lambda^3 - 4\lambda^2 - \lambda + 6,$$

which has roots $\lambda = 1, -2, -3$. These are the eigenvalues.

$\boldsymbol{\lambda = 1.}$ $\begin{pmatrix} -6 & 3 & 3 \\ 0 & -2 & 2 \\ -6 & 5 & 1 \end{pmatrix} \boldsymbol{v} = 0$. We eliminate: $\left.\begin{array}{ccc|c} -6 & 3 & 3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right. \Longrightarrow \left.\begin{array}{ccc|c} -6 & 3 & 3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right.$

The zero row signifies that the original three rows were dependent (they have to be, because the determinant is zero!). We really only have two equations for our three unknowns $v_1, v_2, v_3$. We are free to set, for instance, $v_3 = c$. Then the second equation ($-2v_2 + 2v_3 = 0$) implies $v_2 = c$. Finally, the first equation ($-6v_1 + 3v_2 + 3v_3 = 0$) implies $v_1 = c$. The most general solution to the eigenvector equation therefore is $\boldsymbol{v} = (c, c, c)^T$. Since we don't care about multiples, we choose $\boldsymbol{v} = (1, 1, 1)^T$.

$\boldsymbol{\lambda = -2.}$ $\begin{pmatrix} -3 & 3 & 3 \\ 0 & 1 & 2 \\ -6 & 5 & 4 \end{pmatrix} \boldsymbol{v} = 0$. We eliminate: $\left.\begin{array}{ccc|c} -3 & 3 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right.$ and see that the third equation is redundant.

Let $v_3 = c$. Then, $v_2 = -2c$ (from the second equation). Finally, $-3v_1 + 3v_2 + 3v_3 = 0$ implies $v_1 = -c$. Thus, $\boldsymbol{v} = (-c, -2c, c)^T$ and we choose, for instance, $\boldsymbol{v} = (1, 2, -1)^T$.

$\boldsymbol{\lambda = -3.}$ $\begin{pmatrix} -2 & 3 & 3 \\ 0 & 2 & 2 \\ -6 & 5 & 5 \end{pmatrix} \boldsymbol{v} = 0$. We eliminate: $\left.\begin{array}{ccc|c} -2 & 3 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right.$ and see that the third equation is redundant[17].

Let $v_3 = c$. Then, $v_2 = -c$. Finally, $-2v_1 + 3v_2 + 3v_3 = -2v_1 = 0$ implies $v_1 = 0$. Thus, $\boldsymbol{v} = (0, -c, c)^T$ and we choose, for instance, $\boldsymbol{v} = (0, 1, -1)^T$.

Consequently, we have found solutions $\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^t$, $\boldsymbol{x}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} e^{-2t}$, $\boldsymbol{x}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-3t}$.

They are independent (you could use the Wronskian to check!) as will always be the case if we use our approach appropriately to solve systems with constant coefficients. ♢

As in the case of one-dimensional equations, there are two issues we will have to think and worry about. Firstly, an eigenvalue may be a repeated root of the characteristic polynomial and, secondly, we may encounter complex roots. The case of complex roots is simple to address, the first issue, however, will cause some headaches.

17. With some practice, you can decide right away that the third (or, in fact, any one of the three) is redundant, that is, a linear combination of the other two. You are ready for that shortcut when you know how to explain it!

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