Review. complex numbers $z = x + iy = re^{i\theta}$, conjugate $\bar{z} = x - iy = re^{-i\theta}$ ◇

Note that $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$. In particular, $z\bar{z}$ is always real.

Example 114. $\frac{1}{3-4i} = \frac{3+4i}{(3-4i)(3+4i)} = \frac{3+4i}{3^2+4^2} = \frac{3}{25} + \frac{4}{25}i$ ◇

In general, $\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{|z_2|^2}$. Hence, we can do algebra using complex numbers. For instance, we can do elimination to solve linear equations involving complex numbers.

Note that the real part $\mathrm{Re}(z) = x = \frac{1}{2}(z + \bar{z})$ and the imaginary part $\mathrm{Im}(z) = y = \frac{1}{2i}(z - \bar{z})$ can each be written as a linear combination of z and its conjugate. Since complex solutions to homogeneous linear DEs come in conjugate pairs, we have the following principle.

Theorem 115. If $\boldsymbol{x}(t)$ solves $\boldsymbol{x}' = A(t)\boldsymbol{x}$, where A is $n \times n$ with real entries, then $\mathrm{Re}(\boldsymbol{x}(t))$ and $\mathrm{Im}(\boldsymbol{x}(t))$ are solutions as well.

Example 116. If $\boldsymbol{x}(t) = \begin{pmatrix} 2-3i \\ i \\ 1+i \end{pmatrix} e^{(2+5i)t}$, then we use $e^{(2+5i)t} = e^{2t}(\cos(5t) + i\sin(5t))$ to find

$\mathrm{Re}(\boldsymbol{x}(t)) = e^{2t}\begin{pmatrix} 2\cos(5t) + 3\sin(5t) \\ -\sin(5t) \\ \cos(5t) - \sin(5t) \end{pmatrix}$ and $\mathrm{Im}(\boldsymbol{x}(t)) = e^{2t}\begin{pmatrix} 2\sin(5t) - 3\cos(5t) \\ \cos(5t) \\ \sin(5t) + \cos(5t) \end{pmatrix}$. ◇

Example 117. Find the general solution of $\boldsymbol{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\boldsymbol{x}$.

Solution. The characteristic polynomial $\det\begin{pmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{pmatrix} = (1-\lambda)(-1-\lambda) + 5 = \lambda^2 + 4$ has roots $\pm 2i$.

To find the eigenvector for $\lambda = 2i$, we solve $\begin{pmatrix} 1-2i & -5 \\ 1 & -1-2i \end{pmatrix}\boldsymbol{v} = 0$.

Depending on which row we look at, we can "see" either $\boldsymbol{v}_1 = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}$ or $\boldsymbol{v}_2 = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix}$. At least, we can check that they are both eigenvectors. We can also check that they are just multiples of each other: $\boldsymbol{v}_1 = (1-2i)\boldsymbol{v}_2$.

Alternatively, we could just do elimination: by subtracting $\frac{1}{1-2i} = \frac{1}{5} + \frac{2}{5}i$ times the first row from the second row, $\begin{pmatrix} 1-2i & -5 \\ 0 & 0 \end{pmatrix}\boldsymbol{v} = 0$. Note that we had to get the zero row. Why!? Thus we are free to set $v_2 = c$, and the first equation ($(1-2i)v_1 - 5v_2 = 0$) gives us $v_1 = \frac{5}{1-2i}c = (1+2i)c$. Hence, the most general solution to the eigenvector equation is

$$\boldsymbol{v} = \begin{pmatrix} (1+2i)c \\ c \end{pmatrix},$$

and we observe that $\boldsymbol{v}_2$ is the choice $c = 1$ while $\boldsymbol{v}_1$ is the choice $c = 1 - 2i$.

We do not need to find the eigenvector for $\lambda = -2i$ (it will just be the conjugate of the eigenvector we just found).

Finally, we split the complex solution $\boldsymbol{x} = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix} e^{2it}$ into real and imaginary part:

$\mathrm{Re}(\boldsymbol{x}) = \begin{pmatrix} \cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix}$, $\mathrm{Im}(\boldsymbol{x}) = \begin{pmatrix} \sin(2t) + 2\cos(2t) \\ \sin(2t) \end{pmatrix}$

The general solution is $c_1\begin{pmatrix} \cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2\begin{pmatrix} \sin(2t) + 2\cos(2t) \\ \sin(2t) \end{pmatrix} = \begin{pmatrix} (c_1 + 2c_2)\cos(2t) + (c_2 - 2c_1)\sin(2t) \\ c_1\cos(2t) + c_2\sin(2t) \end{pmatrix}$. ◇

Remark 118. Note that solutions can look different, while being equivalent. For instance, in the previous example, we could have instead chosen the complex solution $\tilde{\boldsymbol{x}} = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix} e^{2it}$. Then, $\mathrm{Re}(\tilde{\boldsymbol{x}}) = \begin{pmatrix} 5\cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix}$, $\mathrm{Im}(\tilde{\boldsymbol{x}}) = \begin{pmatrix} 5\sin(2t) \\ \sin(2t) - 2\cos(2t) \end{pmatrix}$. That these solutions generate the same general solution follows because $\mathrm{Re}(\tilde{\boldsymbol{x}}) = \mathrm{Re}(\boldsymbol{x}) + 2\mathrm{Im}(\boldsymbol{x})$ and $\mathrm{Im}(\tilde{\boldsymbol{x}}) = \mathrm{Im}(\boldsymbol{x}) - 2\mathrm{Re}(\boldsymbol{x})$. Both of these are consequences of $\tilde{\boldsymbol{x}} = (1-2i)\boldsymbol{x}$; can you see that?! ◇

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