Example 119. Solve $\boldsymbol{x}' = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}\boldsymbol{x}$.

Solution. The characteristic polynomial $(4-\lambda)^2+9$ has roots $4 \pm 3i$.

To find the eigenvector for $\lambda = 4+3i$, we solve $\begin{pmatrix} 3i & -3 \\ 3 & 3i \end{pmatrix}\boldsymbol{v} = 0$. We find $\boldsymbol{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

The complex solution $\boldsymbol{x} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(4+3i)t}$ has real part $\operatorname{Re}(\boldsymbol{x}) = e^{4t}\begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix}$ and imaginary part $\operatorname{Im}(\boldsymbol{x}) = e^{4t}\begin{pmatrix} \sin(3t) \\ -\cos(3t) \end{pmatrix}$. The corresponding fundamental matrix is $X(t) = e^{4t}\begin{pmatrix} \cos(3t) & \sin(3t) \\ \sin(3t) & -\cos(3t) \end{pmatrix}$.

[The fundamental matrix satisfies $X' = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} X$. Why?!] ◇

Remark 120. (JustForFun) Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then we easily check that $J^2 = -I$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix. If we think of I as a matrix version of the number 1, then J is a matrix version of the imaginary unit i. More generally, we can then associate a complex number $x+iy$ with the matrix $xI + yJ = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$.
Addition and multiplication work equally on both sides of this identification; why?! Division works equally as well: $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1} = \frac{1}{x^2+y^2}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ corresponds to $(x+iy)^{-1} = \frac{x-iy}{x^2+y^2}$.

The reason that complex numbers can be associated with 2×2 matrices is that multiplying a point in the plane (represented by a complex number $se^{i\varphi}$) with a complex number $re^{i\theta}$ has a geometric meaning (which identifies it as a linear transformation): the point gets scaled by the factor r and rotated by θ (to get $rse^{i(\theta+\varphi)}$).
Since $e^{i\theta}$ is just rotation by θ, the corresponding matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is the rotation matrix that you have probably seen before. Now, you understand why it takes this form! Also, you know that things look simpler if we let complex numbers represent these matrices. That's something people working in computer graphics do!

[This works for 2 dimensions, so is there something[18] like the complex numbers for 3 dimensions? It turns out that the answer is no. However, as discovered by Hamilton, in four dimensions one finds the quaternions $x+iy+jz+kw$ with the rules $i^2=-1$, $j^2=-1$, $k^2=-1$, $ijk=-1$. They satisfy all the usual rules, except that commutativity is lost (and the quaternions are the only such construction). Quaternions are indeed used in computer graphics!]

By the way, with this understanding, we can give a quick (but somewhat unjustified) solution to the previous example (note the special form of the involved matrix). To solve $X' = (4I+3J)X$, where all quantities are 2×2 matrices, we interpret them as complex numbers to get $X(t) = e^{(4+3i)t} = e^{4t}(\cos(3t) + i\sin(3t))$ which we then reinterpret as the matrix $X(t) = e^{4t}\begin{pmatrix} \cos(3t) & -\sin(3t) \\ \sin(3t) & \cos(3t) \end{pmatrix}$. This agrees perfectly with our previous solution (which only differs in the sign for the second column). ◇

Remark 121. (JustForFun) The sequence $(x_0, x_1, x_2, \ldots) = (0,1,1,2,3,5,8,13,21,34,55,89,\ldots)$ is world-famous and known as the Fibonacci sequence. It is defined by $x_{n+1} = x_n + x_{n-1}$ (the recurrence; a discrete analog of a differential equation) and $x_0 = 0$, $x_1 = 1$ (the initial conditions).

Just as we did for differential equations, we can convert the second-order recurrence into a system of first-order equations: $\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_n + x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}$

Iterating, we get $\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2\begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix} = \ldots = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n\begin{pmatrix} x_1 \\ x_0 \end{pmatrix} = A^n\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Note that if $\boldsymbol{v}$ is an eigenvector of A, then $A^n\boldsymbol{v}$ is easy to compute; it is just $A^n\boldsymbol{v} = \lambda^n\boldsymbol{v}$ where λ is the eigenvalue.
The characteristic polynomial of A is $\lambda^2 - \lambda - 1$, and the eigenvalues are $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$ (the golden mean!) and $\lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be the corresponding eigenvectors[19] (which we could compute; but which we will not need for the conclusion we want to make).

Write[20] $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1\boldsymbol{v}_1 + c_1\boldsymbol{v}_2$. Then $A^n\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda_1^n c_1\boldsymbol{v}_1 + \lambda_2^n c_2\boldsymbol{v}_2 \approx \lambda_1^n c_1\boldsymbol{v}_1$ (because $|\lambda_2| < 1$).

Thus, approximately, increasing n by 1 results in multiplication by $\lambda_1 \approx 1.618$.

Indeed, $\frac{x_{n+1}}{x_n} \approx \lambda_1$ is witnessed even by the early examples $\frac{34}{21} \approx 1.61905$, $\frac{55}{34} \approx 1.61765$, $\frac{89}{55} \approx 1.61818$. ◇

18. We are looking for i and j, together with some rules like $i^2=-1$, such that the "numbers" $x+iy+jz$ can be added and multiplied as usual with the important requirement that every element except 0 should be invertible.

19. $\boldsymbol{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$ would do.

20. If we did the calculations with the above choice of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, we would find $c_1 = \frac{5+\sqrt{5}}{10}$, $c_2 = \frac{5-\sqrt{5}}{10}$.

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