Review — currently, all DEs considered are linear

- Homogeneous linear DEs of order n: $Ly = 0$
  - constant coefficients

    using the roots of the characteristic polynomial of L, we have a complete recipe for finding n independent solutions (and we know how to deal with complex and repeated roots)
  - non-constant coefficients

    we have no method for solving such equations (except first-order DEs by integrating factor); however, we know existence of n independent solutions; moreover, if we are handed n prospective solutions, then we can determine whether these are independent solutions (plugging into the DE to check that they actually solve, and then using the Wronskian to check independence)
- Inhomogeneous linear DEs: $Ly = f$

  first of all, we know that if we find a single solution y_p then we get the general solution by adding the solutions of the homogeneous equation
  - constant coefficients plus suitable f (namely, f solves a const coeff eq $\tilde{L}f = 0$)

    by combining the roots of L (the "old" ones) with the roots of $\tilde{L}$ (the "new" ones), we have a recipe to find y_p; namely, since $\tilde{L}Ly_p = 0$, there has to be a y_p that is a combination of the "new" solutions; once we have the shape of y_p with undetermined coefficients, we need to plug into the DE to find these coefficients
  - non-constant coefficients

    once we know the general solution of the homogeneous equation, we can use variation of constants to find y_p; we have only discussed the second-order case, for which we have derived a formula in terms of integrals involving two independent solutions y_1, y_2 of $Ly = 0$; this is one the few (the only?) formulas that you should memorize for the test (deriving takes too long)
- Homogeneous systems of linear DEs: $\boldsymbol{x}' = A(t)\boldsymbol{x}$, where $A(t)$ is an $n \times n$ matrix

  we know that any (linear) DE of order n can be written as a $n \times n$ (linear) first-order system
  - constant coefficients (that is, A does not depend on t)

    we again have a recipe for finding n independent solutions; namely, we find the eigenvalues λ as the roots of the characteristic polynomial $\det(A - \lambda I)$ and then find corresponding eigenvectors $\boldsymbol{v}$; each pair gives us a solution $\boldsymbol{v}e^{\lambda t}$; we do not yet know how to deal with complex and repeated eigenvalues
  - non-constant coefficients (note how our knowledge matches the case of HLDEs of order n)

    we have no method for solving such equations; however, we know existence of n independent solutions; moreover, if we are handed n prospective solutions, then we can determine whether these are independent solutions (plugging into the DE to check that they actually solve, and then using the Wronskian to check independence)
- Mechanical vibrations
  - $mx'' + kx = 0$ describes oscillations of a mass m on a spring with spring constant k

    that's the undamped case; for these, and other oscillations, we know how to determine amplitude and frequency (using that $A\cos(\omega t) + B\sin(\omega t) = \sqrt{A^2 + B^2}\cos(\omega t - \alpha)$)
  - $mx'' + cx' + kx = 0$ models damped motion ($c > 0$ is the damping coefficient)

    solutions can take three different forms: $Ae^{-\rho t}\cos(\omega t - a)$ (underdamped), $Ae^{-\rho_1 t} + Be^{-\rho_2 t}$ (overdamped), or $(A + Bt)e^{-\rho t}$ (critically damped)
  - $mx'' + cx' + kx = f(x)$ models addition of an external force (usually periodic)

    if $c = 0$ then there is the possibility of resonance if natural and external frequency match; if $c > 0$ then we might still have practical resonance; also, if $c > 0$ (and f is periodic), then solutions x split into $x = x_{\text{sp}} + x_{\text{tr}}$, the steady periodic oscillations x_{sp} and the transient motion x_{tr}

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