Review. We now have a recipe to solve $\boldsymbol{x}' = A\boldsymbol{x}$. If A is $n \times n$ (with constant entries), then we can find n independent solutions $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.

In particular, the general solution is $c_1\boldsymbol{x}_1(t) + \ldots + c_n\boldsymbol{x}_n(t) = \Phi(t)\,\boldsymbol{c}$ where $\Phi(t)$ is the fundamental matrix $\Phi = (\boldsymbol{x}_1\ \ldots\ \boldsymbol{x}_n)$. $\diamondsuit$

Let us solve (in a general abstract sense) the IVP $\boldsymbol{x}' = A\boldsymbol{x}$, $\boldsymbol{x}(0) = \boldsymbol{x}_0$.

If $\boldsymbol{x}(t) = \Phi(t)\boldsymbol{c}$, then $\boldsymbol{x}(0) = \Phi(0)\boldsymbol{c} \overset{!}{=} \boldsymbol{x}_0$. We conclude that $\boldsymbol{c} = \Phi(0)^{-1}\boldsymbol{x}_0$. [Why is the matrix $\Phi(0)$ invertible?!]

Hence, $\boldsymbol{x}(t) = \Phi(t)\Phi(0)^{-1}\boldsymbol{x}_0$. Note that $\Phi(t)\Phi(0)^{-1}$ is another fundamental matrix. Why?!

Theorem 129. Let $\Phi(t)$ be any fundamental matrix. Then $\Phi(t)\Phi(0)^{-1} = e^{At}$.

Here, the matrix exponential e^A is defined as $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \ldots$ This is the same Taylor series which we know for the ordinary exponential.

Let us remind ourselves how the Taylor series $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ is connected with the simple differential equation satisfied by the exponential function.

Let us demonstrate, only using the Taylor series, that $x(t) = e^{at}$ solves $x' = ax$. Indeed,

$$x'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\infty} \frac{a^n t^n}{n!} = \sum_{n=1}^{\infty} \frac{a^n n t^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{a^n t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{a^{n+1} t^n}{n!} = a e^{at} = a x(t).$$

- Think about this: every step of this argument works equally well if the number a is replaced by a $n \times n$ matrix A!
- In fact, this shows that e^{At} is a fundamental matrix of $\boldsymbol{x}' = A\boldsymbol{x}$.
- Clearly, $e^{At}|_{t=0} = I$. These two facts together actually prove our theorem above.

Example 130. Find a fundamental matrix for the (very easy) DE $\boldsymbol{x}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\boldsymbol{x}$.

Solution. **(using e^{At})** Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Note that $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. A matrix such that $A^n = 0$, for some n, is called nilpotent. In a such a case, the infinite sum $e^{At} = I + At + A^2\frac{t^2}{2} + \ldots$ is actually finite.

Here, $e^{At} = I + At = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. This is a fundamental matrix.

Solution. **(using generalized eigenvectors)** The characteristic polynomial is λ^2. Hence, $\lambda = 0$ is the only eigenvalue and has multiplicity 2. Solving the eigenvector equation $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\boldsymbol{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we (only) find $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (or any multiple). This means that $\lambda = 0$ has defect 1. To find a generalized eigenvector $\boldsymbol{v}_2$ of rank 2, we solve $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and obtain, for instance, $\boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The corresponding solutions are $\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{x}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix}$. This again reveals $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ as a fundamental matrix.

Solution. **(using nothing)** With $\boldsymbol{x} = (x_1, x_2)^T$, the system can also be written as $x_1' = x_2$, $x_2' = 0$.

The second equation implies $x_2 = c$ for a constant c. Then the first equation shows $x_1 = ct + d$.

Thus $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ct + d \\ c \end{pmatrix} = d\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} t \\ 1 \end{pmatrix}$. Once more, we find $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ as a fundamental matrix. $\diamondsuit$

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