Inhomogeneous linear systems

$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}(t)$ is the general form of a (first-order[25]) inhomogeneous system of linear DEs.

- To solve it, we find a particular solution $\boldsymbol{x}_p(t)$.
- Then, the general solution is $\boldsymbol{x}(t) = \boldsymbol{x}_p(t) + \boldsymbol{x}_c(t)$.

  Here, $\boldsymbol{x}_c(t)$ is the general solution of $\boldsymbol{x}' = A\boldsymbol{x}$.
- Two methods: undetermined coefficients and variation of constants

Example 137. Find the general solution of $\boldsymbol{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 0 \\ -2e^{3t} \end{pmatrix}$.

Solution. From the previous lecture, we know that $\boldsymbol{x}_c = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{4t}$.

We look for a solution of the shape $\boldsymbol{x}_p = \boldsymbol{a}e^{3t}$. To determine the undetermined coefficients, we plug into the DE.

$\boldsymbol{x}_p' = 3\boldsymbol{a}e^{3t} \overset{!}{=} \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\boldsymbol{a}e^{3t} + \begin{pmatrix} 0 \\ -2e^{3t} \end{pmatrix} = \begin{pmatrix} 2a_1 + 3a_2 \\ 2a_1 + a_2 - 2 \end{pmatrix}e^{3t}$

Solving $\begin{pmatrix} 3a_1 \\ 3a_2 \end{pmatrix} = \begin{pmatrix} 2a_1 + 3a_2 \\ 2a_1 + a_2 - 2 \end{pmatrix}$, we find $\boldsymbol{a} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$.

Hence, the general solution is $\begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}e^{3t} + c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{4t}$. ◊

Example 138. Find a particular solution of $\boldsymbol{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 2e^{4t} \\ e^{4t} \end{pmatrix}$.

Solution. Duplication! The 4 ("new" root) from e^{4t} in the inhomogeneous part coincides with an eigenvalue ("old" root). In our particular solution, we therefore include a term $\boldsymbol{a}te^{4t}$. However, that is not enough; as in the case of generalized eigenvectors, we need to include lower order terms as well.

Hence, we look for a solution of the form $\boldsymbol{x}_p = (\boldsymbol{a}t + \boldsymbol{b})\,e^{4t}$. To determine $\boldsymbol{a}$ and $\boldsymbol{b}$, we plug into the differential equation. (Note that $\boldsymbol{b}$ will not be unique: if $\boldsymbol{b}$ works, then so does any $\boldsymbol{b} + c\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Why?!)

$\boldsymbol{x}_p' = (4\boldsymbol{a}t + \boldsymbol{a} + 4\boldsymbol{b})\,e^{4t} \overset{!}{=} \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}(\boldsymbol{a}t + \boldsymbol{b})e^{4t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{4t}$

Equating the coefficients of te^{4t}, we get $4\boldsymbol{a} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\boldsymbol{a}$ or, equivalently, $\begin{pmatrix} 2a_1 - 3a_2 \\ -2a_1 + 3a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Equating the coefficients of e^{4t}, we get $\boldsymbol{a} + 4\boldsymbol{b} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\boldsymbol{b} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ or, equivalently, $\begin{pmatrix} a_1 + 2b_1 - 3b_2 \\ a_2 - 2b_1 + 3b_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Adding the last two equations, we get $a_1 + a_2 = 3$. Together with $2a_1 - 3a_2 = 0$, this gives $\boldsymbol{a} = \begin{pmatrix} 9/5 \\ 6/5 \end{pmatrix}$.

We are left with $2b_1 - 3b_2 = 2 - a_1 = \frac{1}{5}$. We choose $b_2 = 0$, in which case we find $b_1 = \frac{1}{10}$.

In conclusion, we have found the particular solution $\boldsymbol{x}_p = \begin{pmatrix} 9/5t + 1/10 \\ 6/5t \end{pmatrix}e^{4t}$. ◊

Example 139. Find a particular solution of $\boldsymbol{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 2e^{4t} \\ e^{4t} - 2e^{3t} \end{pmatrix}$.

Solution. Note that $\begin{pmatrix} 2e^{4t} \\ e^{4t} + e^{3t} \end{pmatrix} = \begin{pmatrix} 2e^{4t} \\ e^{4t} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ -2e^{3t} \end{pmatrix}$.

Using the particular solutions from the two previous examples, we therefore have the particular solution

$\boldsymbol{x}_p = \begin{pmatrix} 9/5t + 1/10 \\ 6/5t \end{pmatrix}e^{4t} - \frac{1}{2}\begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}e^{3t}$. ◊

25. Recall that any system can be written as a (larger) first-order system.

Armin Straub
astraub@illinois.edu