Review. undetermined coefficients ♢

Example 140. Consider $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}(t)$, where A is a 7×7 matrix with eigenvalues $1\pm 2i$, $1\pm 2i, 0, 3, 3$. For different choices of $\boldsymbol{f}(t)$, we set up $\boldsymbol{x}_p$ with undetermined coefficients.

$\boldsymbol{f}(t)$	"new" roots	$\boldsymbol{x}_p$
$\boldsymbol{g}e^t$	1	$\boldsymbol{a}e^t$
$\boldsymbol{g}$	0	$\boldsymbol{a}t + \boldsymbol{b}$
$\boldsymbol{g}\sin(2t)$	$\pm 2i$	$\boldsymbol{a}\cos(2t) + \boldsymbol{b}\sin(2t)$
$\boldsymbol{g}e^t\sin(2t)$	$1\pm 2i$	$(\boldsymbol{a}_1t^2 + \boldsymbol{a}_2t + \boldsymbol{a}_3)e^t\cos(2t) + (\boldsymbol{a}_4t^2 + \boldsymbol{a}_5t + \boldsymbol{a}_6)e^t\sin(2t)$
$\boldsymbol{g}(t^2+7)e^{3t}$	$3, 3, 3$	$(\boldsymbol{a}_1t^4 + \boldsymbol{a}_2t^3 + \boldsymbol{a}_3t^2 + \boldsymbol{a}_4t + \boldsymbol{a}_5)e^{3t}$
$\boldsymbol{g}(t^2-3t) + \boldsymbol{h}e^t\cos(t)$	$0, 0, 0, 1\pm i$	$(\boldsymbol{a}_1t^3 + \boldsymbol{a}_2t^2 + \boldsymbol{a}_3t + \boldsymbol{a}_4) + \boldsymbol{a}_5e^t\cos(t) + \boldsymbol{a}_6e^t\sin(t)$

It should be remarked that, based on the information on A that we have, the forms for $\boldsymbol{x}_p$ are for the "worst possible" case. If, for instance, the eigenvalue $1\pm 2i$ had no defect, then the form of $\boldsymbol{x}_p$ for $\boldsymbol{f}(t) = \boldsymbol{g}e^t\sin(2t)$ would simplify to $\boldsymbol{x}_p = (\boldsymbol{a}_1t + \boldsymbol{a}_2)e^t\cos(2t) + (\boldsymbol{a}_3t + \boldsymbol{a}_4)e^t\sin(2t)$. Do you see why? ♢

Theorem 141. (variation of constants) The DE $\boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{f}(t)$ is solved by

$$\boldsymbol{x}_p(t) = \Phi(t)\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t.$$

Here, $\Phi(t)$ is any fundamental matrix for $\boldsymbol{x}' = A(t)\boldsymbol{x}$.

Proof. Recall that the general solution of the homogeneous equation $\boldsymbol{x}' = A(t)\boldsymbol{x}$ is $\boldsymbol{x}_c = \Phi(t)\boldsymbol{c}$. We are going to vary the constant $\boldsymbol{c}$ and look for a particular solution of the form $\boldsymbol{x}_p = \Phi(t)\boldsymbol{u}(t)$.

Plugging into the DE, we get

$\boldsymbol{x}_p'(t) = \Phi'(t)\boldsymbol{u}(t) + \Phi(t)\boldsymbol{u}'(t) = A\Phi(t)\boldsymbol{u}(t) + \Phi(t)\boldsymbol{u}'(t) \overset{!}{=} A\boldsymbol{x}_p(t) + \boldsymbol{f}(t) = A\Phi(t)\boldsymbol{u}(t) + \boldsymbol{f}(t).$

For the first equality, we used the matrix version of the usual product rule (which holds since differentiation is defined entry-wise). For the second equality, we used $\Phi' = A\Phi$.

Subtracting $A\Phi\boldsymbol{u}$, we see that $\boldsymbol{x}_p = \Phi(t)\boldsymbol{u}(t)$ is a solution if and only if $\Phi(t)\boldsymbol{u}'(t) = \boldsymbol{f}(t)$.

Hence, $\boldsymbol{u}'(t) = \Phi(t)^{-1}\boldsymbol{f}(t)$ and it only remains to integrate. □

Example 142. Find a particular solution of $\boldsymbol{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 0 \\ -2e^{3t} \end{pmatrix}$.

Solution. From previous examples, we know that $\Phi(t) = \begin{pmatrix} e^{-t} & 3e^{4t} \\ -e^{-t} & 2e^{4t} \end{pmatrix}$.

Using $\det\Phi = 5e^{3t}$, we find $\Phi(t)^{-1} = \frac{1}{5}\begin{pmatrix} 2e^t & -3e^t \\ e^{-4t} & e^{-4t} \end{pmatrix}$.

Hence, $\Phi(t)^{-1}\boldsymbol{f}(t) = \frac{2}{5}\begin{pmatrix} 3e^{4t} \\ -e^{-t} \end{pmatrix}$ and $\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t = \frac{2}{5}\begin{pmatrix} 3/4e^{4t} \\ e^{-t} \end{pmatrix}$.

By variation of constants, $\boldsymbol{x}_p(t) = \Phi(t)\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t = \begin{pmatrix} e^{-t} & 3e^{4t} \\ -e^{-t} & 2e^{4t} \end{pmatrix}\frac{2}{5}\begin{pmatrix} 3/4e^{4t} \\ e^{-t} \end{pmatrix} = \frac{2}{5}\begin{pmatrix} 15/4 \\ 5/4 \end{pmatrix}e^{3t} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}e^{3t}$.

Note that this matches the result we obtained in Example 137.

By the way, why do we not need to be careful about the constants of integration? ♢

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