Review. Variation of constants: $\boldsymbol{x}_p(t) = \Phi(t)\int \Phi(t)^{-1}\boldsymbol{f}(t)\mathrm{d}t$ solves $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}(t)$

Here, $\Phi(t)$ is any fundamental matrix of $\boldsymbol{x}' = A\boldsymbol{x}$. ♢

In the special case that $\Phi(t) = e^{At}$, some things become easier. For instance, $\Phi(t)^{-1} = e^{-At}$. Also, we can just write down solutions to IVPs:

- $\boldsymbol{x}' = A\boldsymbol{x}$, $\boldsymbol{x}(0) = \boldsymbol{x}_0$ has (unique) solution $\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0$.
- $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}(t)$, $\boldsymbol{x}(0) = \boldsymbol{x}_0$ has (unique) solution $\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0 + e^{At}\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s$.

Example 143. Suppose that the matrix A satisfies $e^{At} = \begin{pmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{pmatrix}$.

- Solve $\boldsymbol{x}' = A\boldsymbol{x}$, $\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Solution. $\boldsymbol{x}(t) = e^{At}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2e^{2t} + 3e^{3t} \\ -e^{2t} + 3e^{3t} \end{pmatrix}$.

- Solve $\boldsymbol{x}' = A\boldsymbol{x} + \begin{pmatrix} 0 \\ 2e^t \end{pmatrix}$, $\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Solution. $\boldsymbol{x}(t) = e^{At}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{At}\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s$. We compute:

$$\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s = \int_0^t \begin{pmatrix} 2e^{-2s} - e^{-3s} & -2e^{-2s} + 2e^{-3s} \\ e^{-2s} - e^{-3s} & -e^{-2s} + 2e^{-3s} \end{pmatrix}\begin{pmatrix} 0 \\ 2e^s \end{pmatrix}\mathrm{d}s = \int_0^t \begin{pmatrix} -4e^{-s} + 4e^{-2s} \\ -2e^{-s} + 4e^{-2s} \end{pmatrix}\mathrm{d}s = \begin{pmatrix} 4e^{-t} - 2e^{-2t} - 2 \\ 2e^{-t} - 2e^{-2t} \end{pmatrix}$$

Hence, $e^{At}\int_0^t e^{-As}\boldsymbol{f}(s)\mathrm{d}s = \begin{pmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{pmatrix}\begin{pmatrix} 4e^{-t} - 2e^{-2t} - 2 \\ 2e^{-t} - 2e^{-2t} \end{pmatrix} = \begin{pmatrix} 2e^t - 4e^{2t} + 2e^{3t} \\ -2e^{2t} + 2e^{3t} \end{pmatrix}$.

Finally, $\boldsymbol{x}(t) = \begin{pmatrix} -2e^{2t} + 3e^{3t} \\ -e^{2t} + 3e^{3t} \end{pmatrix} + \begin{pmatrix} 2e^t - 4e^{2t} + 2e^{3t} \\ -2e^{2t} + 2e^{3t} \end{pmatrix} = \begin{pmatrix} 2e^t - 6e^{2t} + 5e^{3t} \\ -3e^{2t} + 5e^{3t} \end{pmatrix}$.

- What is A?

Solution. Like any fundamental matrix, e^{At} satisfies $\frac{\mathrm{d}}{\mathrm{d}t}e^{At} = Ae^{At}$.

Hence, $A = \left[\frac{\mathrm{d}}{\mathrm{d}t}e^{At}\right]_{t=0} = \left[\begin{pmatrix} 4e^{2t} - 3e^{3t} & -4e^{2t} + 6e^{3t} \\ 2e^{2t} - 3e^{3t} & -2e^{2t} + 6e^{3t} \end{pmatrix}\right]_{t=0} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$. ♢

Example 144. Three brine tanks T_1, T_2, T_3. This is Example 5.6.2 in the book.

T_1 contains 20gal water with 10lb salt, T_2 40gal pure water, T_3 50gal water with 30lb salt.

T_1 is filled with 10gal/min water with 2lb/gal salt. 10gal/min well-mixed solution flows out of T_1 into T_2. Also, 10gal/min well-mixed solution flows out of T_2 into T_3. Finally, 10gal/min well-mixed solution is leaving T_3. How much salt is in the tanks after t minutes?

Solution. Let $x_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min).

In time interval $[t, t + \Delta t]$:

$\Delta x_1 \approx 10 \cdot 2 \cdot \Delta t - 10\frac{x_1}{20} \cdot \Delta t$, so $x_1' = 20 - \frac{1}{2}x_1$. Also, $x_1(0) = 10$.

$\Delta x_2 \approx 10 \cdot \frac{x_1}{20} \cdot \Delta t - 10\frac{x_2}{40} \cdot \Delta t$, so $x_2' = \frac{1}{2}x_1 - \frac{1}{4}x_2$. Also, $x_2(0) = 0$.

$\Delta x_3 \approx 10 \cdot \frac{x_2}{40} \cdot \Delta t - 10\frac{x_3}{50} \cdot \Delta t$, so $x_3' = \frac{1}{4}x_2 - \frac{1}{5}x_3$. Also, $x_3(0) = 30$.

Using matrix notation and writing $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, this is $\boldsymbol{x}' = \begin{pmatrix} -1/2 & 0 & 0 \\ 1/2 & -1/4 & 0 \\ 0 & 1/4 & -1/5 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{x}(0) = \begin{pmatrix} 10 \\ 0 \\ 30 \end{pmatrix}$.

We can solve this IVP! (Details in our book.)

Here, we content ourselves with a particular solution (and ignoring the initial conditions). Undetermined coefficients tells us that there is a solution of the form $\boldsymbol{x}_p(t) = \boldsymbol{a}$. Of course, we can find $\boldsymbol{a}$ by plugging into the differential equation. However, noticing that, for a constant solution, each tank has to have a concentration of 2lb/gal of salt, we find $\boldsymbol{x}_p = (40, 80, 100)$ without calculation. ♢

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