Fourier series

Definition 145. Let $L>0$. $f(t)$ is L-periodic if $f(t+L)=f(t)$ for all t. The smallest such L is called "the" period of f.

Example 146. The period of $\cos(nt)$ is $2\pi/n$. ◇

Example 147. The trigonometric functions $\cos(nt), \sin(nt)$ are 2π-periodic. And so are all their linear combinations. (In other words, 2π-periodic functions form a vector space.) ◇

The following amazing fact is saying that any 2π-periodic function can be written as a sum of cosines and sines.

Theorem 148. Every* 2π-periodic function f can be written as a Fourier series[26]

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(nt)+b_n\sin(nt)).$$

Technical detail*: f needs to be, e.g., piecewise smooth.
Also, if t is a discontinuity, then the Fourier series converges to the average $\frac{f(t^-)+f(t^+)}{2}$.

The Fourier coefficients a_n, b_n are unique and can be computed as

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos(nt)\mathrm{d}t, \qquad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin(nt)\mathrm{d}t.$$

Example 149. Find the Fourier series of the 2π-periodic function $f(t)$ defined by

$$f(t)=\begin{cases}-1, & \text{for } t\in(-\pi,0),\\ +1, & \text{for } t\in(0,\pi),\\ 0, & \text{for } t=-\pi,0,\pi.\end{cases}$$

Solution. We compute $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\mathrm{d}t=0$, as well as

$$\begin{aligned}a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos(nt)\mathrm{d}t=\frac{1}{\pi}\left[-\int_{-\pi}^{0}\cos(nt)\mathrm{d}t+\int_0^{\pi}\cos(nt)\mathrm{d}t\right]=0\\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin(nt)\mathrm{d}t=\frac{1}{\pi}\left[-\int_{-\pi}^{0}\sin(nt)\mathrm{d}t+\int_0^{\pi}\sin(nt)\mathrm{d}t\right]=\frac{2}{\pi n}[1-\cos(n\pi)]\\ &= \frac{2}{\pi n}[1-(-1)^n]=\begin{cases}\frac{4}{\pi n} & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even}\end{cases}.\end{aligned}$$

In conclusion, $f(t)=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4}{\pi n}\sin(nt)=\frac{4}{\pi}\left(\sin(t)+\frac{1}{3}\sin(3t)+\frac{1}{5}\sin(5t)+\ldots\right)$. ◇

Remark 150. (JustForFun) Set $t=\frac{\pi}{2}$ in the Fourier series we just computed, to get Leibniz' series[27] $\pi=4[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots]$; which brings us back to our April fool's derivation of $\pi=4$. The trouble boils down to the fact that we would like to conclude that the convergence of functions $f_n\to f$ implies that their arc length $L(f_n)$ converges to $L(f)$. That's a natural instinct (and would be true if L is continuous). However, arc length depends on the derivative (remember its formula?!), and $f_n\to f$ does not[28] necessarily imply $f_n'\to f'$. ◇

26. Another common way to write Fourier series is $f(t)=\sum_{n=-\infty}^{\infty}c_n e^{int}$. These two ways are equivalent; we can convert between them using Euler's identity $e^{int}=\cos(nt)+i\sin(nt)$.

27. For such an alternating series, the error made by stopping at the term $1/n$ is on the order of $1/n$. To compute the 768 digits of π to get to the Feynman point, we would (roughly) need $1/n<10^{-768}$, or $n>10^{768}$. That's a lot of terms! (Roger Penrose, for instance, estimates that there are about 10^{80} atoms in the observable universe.)

28. In other words, taking the derivative of a function is not a continuous operator! That's a subject for functional analysis, which studies spaces of functions as well as operators between such spaces.

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