Sketch of Lecture 41

Thu, 04/10/2014

Review. Fourier series ◇

Fourier series and differential equations

Let us revisit the inhomogeneous equation $mx'' + kx = F(t)$ describing the motion of a mass m on a spring with spring constant k under the influence of an external force $F(t)$.

Recall that, when $F = 0$ (the complementary homogeneous equation), then the solutions are combinations of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$, where $\omega_0 = \sqrt{k/m}$ is the natural frequency.
We have solved equations like $mx'' + kx = \sin(\omega t)$. A crucial insight was that the case $\omega = \omega_0$ (overlapping roots) is special and corresponds to resonance.

We are now going to allow any periodic force $F(t)$, and solve the equation by using the Fourier series for $F(t)$. The same approach works likewise for linear equations of higher order, or even systems of equations.

Example 158. Find a particular solution of $2x'' + 32x = F(t)$, with $F(t) = \begin{cases} 10 & \text{if } t \in (0,1) \\ -10 & \text{if } t \in (1,2) \end{cases}$, extended 2-periodically.

Solution. Step A: From the previous classes, we already know $F(t) = 10 \sum_{n \text{ odd}} \frac{4}{\pi n} \sin(\pi n t)$.
Step B: We next solve the equation $2x'' + 32x = \sin(\pi n t)$ for $n = 1, 3, 5, \ldots$. First, we note that the external frequency is πn, which is never equal to the natural frequency $\omega_0 = 4$. Hence, there exists a particular solution of the form $x_p = A\cos(\pi n t) + B\sin(\pi n t)$. To determine the coefficients A, B, we plug into the DE. Noting that $x_p'' = -\pi^2 n^2 x_p$ (why?!), we get

$$2x_p'' + 32x_p = (32 - 2\pi^2 n^2)(A\cos(\pi n t) + B\sin(\pi n t)) \overset{!}{=} \sin(\pi n t).$$

We conclude $A = 0$ and $B = \frac{1}{32 - 2\pi^2 n^2}$, so that $x_p = \frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}$.
Step C: We combine the particular solutions found in the previous step, to see that

$$2x'' + 32x = 10 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(\pi n t) \quad \text{is solved by} \quad x_p = 10 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}.$$

Note that $x_p(t) = 1.038\sin(\pi t) - 0.029\sin(3\pi t) - 0.0055\sin(5\pi t) - \ldots$ is well approximated by the first two terms. Indeed, the amplitude of x_p is about $1.038 + 0.029$ [first two terms have a maximum at $t = 1/2$]. ◇

Example 159. Find a particular solution of $2x'' + 32x = F(t)$, with $F(t)$ the 2π-periodic function such that $F(t) = 10t$ for $t \in (-\pi, \pi)$.

Solution. Step A: The Fourier series of $F(t)$ is $F(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{20}{n} \sin(nt)$. [Exercise!]
Step B: We next solve the equation $2x'' + 32x = \sin(nt)$ for $n = 1, 2, 3, \ldots$. Note, however, that (pure) resonance does occur for $n = 4$, so we need to treat that case separately. If $n \neq 4$ then we find, as in the previous example, that $x_p = \frac{\sin(nt)}{32 - 2n^2}$. [See how this fails for $n = 4$!]
For $2x'' + 32x = \sin(4t)$, we begin with $x_p = At\cos(4t) + Bt\sin(4t)$. Then $x_p' = (A + 4Bt)\cos(4t) + (B - 4At)\sin(4t)$, and $x_p'' = (8B - 16At)\cos(4t) + (-8A - 16Bt)\sin(4t)$. Plugging into the DE, we get $2x_p'' + 32x_p = 16B\cos(4t) - 16A\sin(4t) \overset{!}{=} \sin(4t)$, and thus $B = 0$, $A = -\frac{1}{16}$. So, $x_p = -\frac{1}{16} t\cos(4t)$.
Step C: We combine the particular solutions to get

$$2x'' + 32x = -5\sin(4t) + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty} (-1)^{n+1} \frac{20}{n} \sin(nt) \quad \text{is solved by} \quad x_p = \frac{5}{16} t\cos(4t) + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty} (-1)^{n+1} \frac{20}{n} \frac{\sin(nt)}{32 - 2n^2}.$$

As in the previous example, this solution cannot really be simplified. Make some plots to appreciate the dominating character of the term resulting from resonance! ◇

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