The Laplace transform

Definition 162. The Laplace transform of a function $f(t)$, $t \geqslant 0$, is defined as the new function

$$F(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t.$$

We also write $\mathcal{L}(f(t)) = F(s)$.

Note that, in order for the integral to exist, $f(t)$ should be, say, piecewise continuous and of at most exponential growth. That's true for most of the functions, we are interested in (and we will not dwell on this issue).

$f(t)$	$F(s)$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
e^{at}	$\frac{1}{s-a}$
1	$\frac{1}{s}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$

Example 163.

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at} \mathrm{d}t = \int_0^\infty e^{(a-s)t} \mathrm{d}t = \left[\frac{1}{a-s} e^{(a-s)t}\right]_{t=0}^\infty = 0 - \frac{1}{a-s} = \frac{1}{s-a}$$

Note that we needed $a - s < 0$ in order for the integral to converge. Hence the Laplace transform has domain $s > a$. (During this introduction, we will not care too much about these technical details.) ◇

Example 164. The Laplace transform is linear:

$$\mathcal{L}(c_1 f_1(t) + c_2 f_2(t)) = \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t)) \mathrm{d}t = c_1 \int_0^\infty e^{-st} f_1(t) \mathrm{d}t + c_2 \int_0^\infty e^{-st} f_2(t) \mathrm{d}t,$$

which equals $c_1 F_1(s) + c_2 F_2(s)$. ◇

Example 165. By Euler's identity, $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$. Hence, by linearity,

$$\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos(\omega t)) + i \mathcal{L}(\sin(\omega t)).$$

On the other hand,

$$\mathcal{L}(e^{i\omega t}) = \frac{1}{s - i\omega} = \frac{s + i\omega}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2} + i \frac{\omega}{s^2 + \omega^2}.$$

Matching real and imaginary parts, gives $\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$ and $\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$. ◇

Example 166. Using integration by parts,

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} f'(t) \mathrm{d}t = [e^{-st} f(t)]_{t=0}^\infty + \int_0^\infty s e^{-st} f(t) \mathrm{d}t = sF(s) - f(0).$$

In order to obtain the Laplace transform of higher derivatives, we can iterate. For instance,

$$\mathcal{L}(f''(t)) = s\mathcal{L}(f'(t)) - f'(0) = s[sF(s) - f(0)] - f'(0) = s^2 F(s) - sf(0) - f'(0).$$ ◇

Example 167. Consider the (very simple) IVP $x'(t) - 2x(t) = 0$, $x(0) = 7$. [Of course, $x(t) = 7e^{2t}$.]

$$\mathcal{L}(x'(t) - 2x(t)) = \mathcal{L}(x'(t)) - 2\mathcal{L}(x(t)) = sX(s) - x(0) - 2X(s) = (s-2)X(s) - 7 = 0.$$

This is an algebraic equation for $X(s)$. It follows that $X(s) = \frac{7}{s-2}$. By inverting the Laplace transform (which is possible!), we conclude that $x(t) = 7e^{2t}$. ◇

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