Review. Eigenvalues and eigenfunctions of $y''+\lambda y=0,\ y(0)=0,\ y(L)=0.$ ♢

Example 186. Suppose that a rod of length L is compressed by a force P. We model the shape of the rod by a function $y(x)$ on the interval $[0,L]$. The theory of elasticity predicts that, under some simplifying assumptions, y should satisfy $EIy''+Py=0,\ y(0)=0,\ y(L)=0.$

Here, EI is a constant modeling the inflexibility of the rod (E, known as Young's modulus, depends on the material, and I depends on the shape of cross-sections (it is the area moment of inertia)).

In other words, $y''+\lambda y=0,\ y(0)=0,\ y(L)=0$, with $\lambda=\frac{P}{EI}$. The fact that there is no nonzero solution unless $\lambda=\left(\frac{\pi n}{L}\right)^2$ for some $n=1,2,3,\ldots$, means that buckling can only occur if $P=\left(\frac{\pi n}{L}\right)^2 EI$. In particular, no buckling occurs for forces less than $\frac{\pi^2 EI}{L^2}$. This is known as the critical load (or Euler load) of the rod. ♢

The heat equation

We wish to describe one-dimensional heat flow[33]. Let $u(x,t)$ describe the temperature at time t at position x. If we model a heated rod of length L, then $x\in[0,L]$.

Note that $u(x,t)$ depends on two variables. When taking derivatives, we will use the notations $u_t=\frac{\partial}{\partial t}u$ and $u_{xx}=\frac{\partial^2}{\partial t^2}u$ for first and higher derivatives.

Experience tells us that heat flows from warmer to cooler areas and has an averaging effect. Make a sketch of some temperature profile $u(x,t)$ for fixed t. Then as t increases, we expect maxima (where $u_{xx}<0$) of that profile to flatten out (so $u_t<0$); similarly, minima (where $u_{xx}>0$) should go up (so $u_t>0$). The simplest relationship between u_t and u_{xx} which conforms with our expectation is $u_t=ku_{xx}$, with $k>0$. That's the heat equation.

Note that the heat equation is a linear and homogeneous partial differential equation.

In particular, the principle of superposition holds: if u_1 and u_2 solve the heat equation, then so does $c_1u_1+c_2u_2$.

Remark 187. In higher dimensions, the heat equation takes the form $u_t=k(u_{xx}+u_{yy})$ or $u_t=k(u_{xx}+u_{yy}+u_{zz})$. Note that $\Delta u=u_{xx}+u_{yy}+u_{zz}$ is the Laplace operator[34]. ♢

Example 188. Note that $u(x,t)=ax+b$ solves the heat equation. ♢

Let us think about what is needed to describe a unique solution of the heat equation.

Initial condition at $t=0$: $u(x,0)=f(x)$ [temperature distribution at time $t=0$]
Boundary condition at $x=0$ and $x=L$: [heat only enters/exits at boundary]
for instance, $u(0,t)=u(L,t)=0$ [by adding $ax+b$ to u this covers any constant boundary values]
[another option are boundary conditions like $u_x(0,t)=u_x(L,t)=0$, which would model insulated ends]

Example 189. To get a feeling, let us find some other solutions to $u_t=u_{xx}$ (for starters, $k=1$).

For instance, $u(x,t)=e^te^x$ is a solution. Not a very interesting one for modeling heat flow because it increases exponentially in time.
More interesting are $u(x,t)=e^{-t}\cos(x)$ and $u(x,t)=e^{-t}\sin(x)$. More generally, $e^{-n^2t}\cos(nx)$ and $e^{-n^2t}\sin(nx)$ are solutions. This actually reveals a strategy for solving the heat equation with conditions such as $u_t=u_{xx}$, $u(0,t)=0$, $u(L,t)=0$, $u(x,0)=f(x)$.
Namely, the solutions $u_n(x,t)=e^{-n^2t}\sin(nx)$ all satisfy $u(0,t)=0$, $u(\pi,t)=0$. On the other hand, $u_n(x,0)=\sin(nx)$. To find $u(x,t)$ such that $u(x,0)=f(x)$, we thus only need to write $f(x)$ as a Fourier (sine) series. ♢

33. If this sounds very specialized, it might help to know that the heat equation is also used, for instance, in probability (Brownian motion), financial math (Black-Scholes), or chemical processes (diffusion equation).

34. The Laplacian Δu is also often written as $\Delta u=\nabla^2 u$. The operator $\nabla=(\partial/\partial x,\partial/\partial y)$ is pronounced "nabla" (Greek for a certain harp) or "del" (Persian for heart), and ∇^2 is short for the inner product $\nabla\cdot\nabla$.

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