Review. heat equation ◇

Example 190. Find the unique solution to:
$$\begin{aligned} &u_t = k u_{xx} && \text{(PDE)}\\ &u(0,t) = u(L,t) = 0 && \text{(BC)}\\ &u(x,0) = f(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

Solution. We will first look for simple solutions of PDE+BC (and then we plan to take a superposition of such solutions that satisfies IC as well). Namely, we look for solutions $u(x,t) = X(x)T(t)$. This approach is called separation of variables and it is crucial for solving other PDEs as well.

Plugging into the PDE, we get $X(x)T'(t) = kX''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)}$. Note that the two sides cannot depend on x (because the right-hand side doesn't) and they cannot depend on t (because the left-hand side doesn't). Hence, they have to be constant. Let's call this constant $-\lambda$. Then, $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{const} =: -\lambda$.

We thus have $X'' + \lambda X = 0$ and $T' + \lambda k T = 0$.

Consider the BC. $u(0,t) = X(0)T(t) = 0$ implies $X(0) = 0$ (because otherwise $T(t) = 0$ for all t, which would mean that $u(x,t)$ is the dull zero solution). Likewise, $u(L,t) = X(L)T(t) = 0$ implies $X(L) = 0$.

So X solves $X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$. We know that, up to multiples, the only nonzero solutions are the eigenfunctions $X(x) = \sin\left(\frac{\pi n}{L} x\right)$ corresponding to the eigenvalues $\lambda = \left(\frac{\pi n}{L}\right)^2$, $n = 1, 2, 3 \ldots$

On the other hand, T solves $T' + \lambda k T = 0$, and hence $T(t) = e^{-\lambda k t} = e^{-\left(\frac{\pi n}{L}\right)^2 k t}$.

Taken together, we have the solutions $u_n(x,t) = e^{-\left(\frac{\pi n}{L}\right)^2 k t} \sin\left(\frac{\pi n}{L} x\right)$ solving PDE+BC.

We wish to combine these in such a way that IC holds. At $t = 0$, $u_n(x,0) = \sin\left(\frac{\pi n}{L} x\right)$. All of these are $2L$-periodic. Hence, we extend $f(x)$, which is only given on $(0,L)$, to an odd $2L$-periodic function. By making it odd, its Fourier series will only involve sine terms: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{L} x\right)$.

Consequently, PDE+BC+IC is solved by $u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{\pi n}{L}\right)^2 k t} \sin\left(\frac{\pi n}{L} x\right)$. ◇

Example 191. Find the unique solution to:
$$\begin{aligned} &u_t = u_{xx}\\ &u(0,t) = u(1,t) = 0\\ &u(x,0) = 1, \quad x \in (0,1) \end{aligned}$$

Solution. This is the case $k = 1$, $L = 1$ and $f(x) = 1$, $x \in (0,1)$, of the previous example.

In the final step, we extend $f(x)$ to the 2-periodic odd function of Example 152. In particular, we have already computed that the Fourier series is $f(x) = \sum_{n=1, n \text{ odd}}^{\infty} \frac{4}{\pi n} \sin(\pi n x)$.

Hence, $u(x,t) = \sum_{n=1, n \text{ odd}}^{\infty} \frac{4}{\pi n} e^{-\pi^2 n^2 t} \sin(\pi n x)$. ◇

The boundary conditions in the next example model insulated ends.

Example 192. Find the unique solution to:
$$\begin{aligned} &u_t = k u_{xx} && \text{(PDE)}\\ &u_x(0,t) = u_x(L,t) = 0 && \text{(BC)}\\ &u(x,0) = f(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

Solution. We proceed as before and look for solutions $u(x,t) = X(x)T(t)$. Plugging into the PDE, we get $X(x)T'(t) = kX''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{const} =: -\lambda$. We thus have $X'' + \lambda X = 0$ and $T' + \lambda k T = 0$.

From the BC $u_x(0,t) = X'(0)T(t) = 0$ we get $X'(0) = 0$. Likewise, $u_x(L,t) = X'(L)T(t) = 0$ implies $X'(L) = 0$.

So X solves $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(L) = 0$. It is left as an exercise (DO IT!!) that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x) = \cos\left(\frac{\pi n}{L} x\right)$ corresponding to $\lambda = \left(\frac{\pi n}{L}\right)^2$, $n = 0, 1, 2, 3 \ldots$

As before, T solves $T' + \lambda k T = 0$, and hence $T(t) = e^{-\lambda k t} = e^{-\left(\frac{\pi n}{L}\right)^2 k t}$.

Taken together, we have the solutions $u_n(x,t) = e^{-\left(\frac{\pi n}{L}\right)^2 k t} \cos\left(\frac{\pi n}{L} x\right)$ solving PDE+BC.

We wish to combine these in such a way that IC holds. At $t = 0$, $u_n(x,0) = \cos\left(\frac{\pi n}{L} x\right)$. All of these are $2L$-periodic. Hence, we extend $f(x)$, which is only given on $(0,L)$, to an even $2L$-periodic function. By making it even, its Fourier series will only involve cosine terms: $f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\left(\frac{\pi n}{L} x\right)$.

So, PDE+BC+IC is solved by $u(x,t) = \frac{a_0}{2} u_0(x,t) + \sum_{n=1}^{\infty} a_n u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{\pi n}{L}\right)^2 k t} \cos\left(\frac{\pi n}{L} x\right)$. ◇

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