Net ID: **Name:**

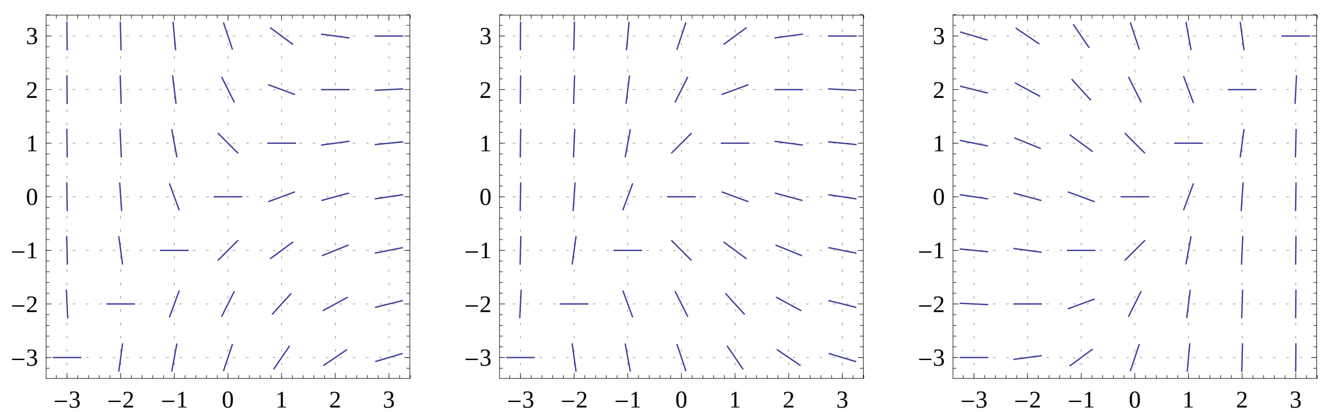
Midterm #1

MATH 286 — Differential Equations Plus
Thursday, February 13

- No notes, personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.

Good luck!

Problem 1. (5 points) Circle the slope field below which belongs to the differential equation $e^x y' = x - y$.



Problem 2. (20 points) Find the general solution to the differential equation $y^{(5)} - 4y^{(4)} + 4y^{(3)} = 0$.

Solution. This is a linear homogeneous equation with constant coefficients. The characteristic polynomial is $r^5 - 4r^4 + 4r^3 = r^3(r-2)^2$.

Hence, the general solution is $c_1 + c_2 x + c_3 x^2 + (c_4 + c_5 x)e^{2x}$. □

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Problem 3. (20 points) Solve the initial value problem

$$(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x}+xy=\frac{1}{\sqrt{x^2+1}}, \qquad y(0)=1.$$

Solution. Dividing by x^2+1, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{x}{x^2+1}\,y=\frac{1}{(x^2+1)^{3/2}},$$

which is a linear equation with integrating factor

$$e^{\int \frac{x}{x^2+1}\mathrm{d}x}=e^{\frac{1}{2}\ln(x^2+1)}=(x^2+1)^{1/2}.$$

Multiply through by this integrating factor, we obtain

$$(x^2+1)^{1/2}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{x}{(x^2+1)^{1/2}}\,y=\frac{\mathrm{d}}{\mathrm{d}x}\left((x^2+1)^{1/2}\,y\right)=\frac{1}{x^2+1}.$$

Integrating both sides with respect to x yields

$$(x^2+1)^{1/2}\,y(x)=\arctan(x)+C.$$

Since $y(0)=1$, we find $C=1$. Therefore,

$$y(x)=\frac{\arctan(x)+1}{(x^2+1)^{1/2}}.$$

□

Problem 4. (20 points) The time rate of change of a rabbit population P is proportional to the square root of P. At time $t=0$, the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be after two months?

Solution. $\frac{\mathrm{d}P}{\mathrm{d}t}=k\sqrt{P}$ and $P(0)=100$, $P'(0)=20$. The problem asks for $P(2)$.

At $t=0$, we have $20=P'(0)=k\sqrt{P(0)}=10k$. Hence $k=2$.

By separation of variables, we find $2\sqrt{P}=2t+C$. From $P(0)=100$, $C=20$. Thus, $P=(t+10)^2$.

We conclude that there are $P(2)=12^2=144$ rabbits after two months. □

Problem 5. (20 points) For each $c \geqslant 0$, let $y_c(x) = \begin{cases} x^3, & \text{if } x < 0, \\ 0, & \text{if } 0 \leqslant x \leqslant c, \\ (x-c)^3, & \text{if } x > c. \end{cases}$

(a) Sketch the graph of $y_c(x)$ for $c = 3/2$.

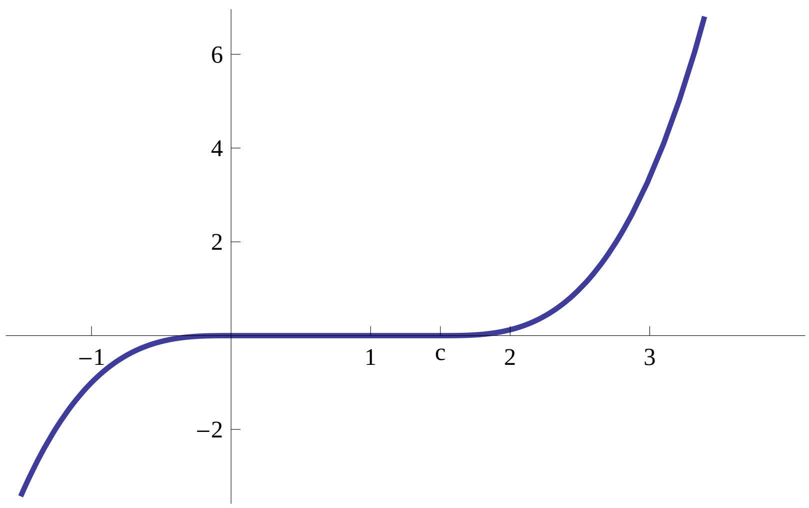
(b) Show that, for all $c \geqslant 0$, y_c is a solution to the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3y^{2/3}, \quad y(0) = 0.$$

(c) Explain why (b) does not contradict the theorem on existence and uniqueness for solutions to initial value problems.

Solution.

(a) For $c = 3/2$, the graph looks as follows.



(b) Clearly, $y_c(0) = 0$. Moreover, since $\frac{\mathrm{d}}{\mathrm{d}x} x^3 = 3x^2 = 3(x^3)^{2/3}$, $\frac{\mathrm{d}}{\mathrm{d}x}(x-c)^3 = 3(x-c)^2 = 3[(x-c)^3]^{2/3}$, and all the left and right derivatives at the appropriate endpoints match, $y_c(x)$ satisfies the DE $y' = 3\,y^{2/3}$.

(c) Note that (b) demonstrates that there are infinitely many solutions to the IVP. Thus we do not have uniqueness.

Write the DE as $y' = f(x, y)$ with $f(x, y) = 3y^{2/3}$ which is continuous around the point $(0, 0) = (0, y(0))$. However, $\frac{\partial}{\partial y} f(x, y) = 2y^{-1/3}$ is not continuous around the point $(0, 0)$. Our theorem on existence and uniqueness therefore only guarantees existence but not uniqueness for this particular IVP. □

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Problem 6. (20 points) Find a general solution to the differential equation

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} - x^2 - y^2 - 3xy = 0.$$

Solution. Dividing through by x^2, the equation becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + (y/x)^2 + 3\,(y/x).$$

Making the homogeneous substitution $v = y/x$, we have $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}v}{\mathrm{d}x} + v = 1 + v^2 + 3v$, which gives

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 + 2\,v + 1 = (v+1)^2.$$

By separation of variables, we get $\frac{\mathrm{d}v}{(v+1)^2} = \frac{\mathrm{d}x}{x}$ (note that we just lost the singular solution $v = -1$ which corresponds to the solution $y = -x$ of the original equation). Integrating, we have $-(v+1)^{-1} = \ln|x| + C$ and hence

$$y(x) = -\frac{x}{\ln|x| + C} - x.$$ □

Problem 7. (5 points) Consider the differential equation *Hint:* Do not attempt to solve the DE.

$$y' = y^4 + x^4 + 1.$$

Is it possible that there exists a solution with the property that $\lim_{x \to \infty} y(x) = -\infty$? Why, or why not?

Solution. It is impossible. It follows from the differential equation that $y'(x) > 0$ for all x. In other words, any solution $y(x)$ is an increasing function, and thus cannot approach $-\infty$ for large x. □

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