Preparing Midterm #2

MATH 286 — Differential Equations Plus
March 13, 2014

Problem 1. Solve $y'' + 2y' + y = 2e^{2x} + e^{-x}$, $y(0) = -1$, $y'(0) = 2$.

Solution. The characteristic equation for the associated homogeneous DE has roots $-1, -1$ (the "old" roots). The right-hand side solves a DE whose characteristic equation has roots $-1, 2$ (the "new" roots).

Hence, there is a particular solution of the form $y_p = Ae^{2x} + Bx^2e^{-x}$. To find A, B we plug into the DE. [...] We find $A = \frac{2}{9}$ and $B = \frac{1}{2}$.

Particular solution: $y_p = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x}$

General solution: $y = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x} + c_1e^{-x} + c_2xe^{-x}$

Now, we use the initial values [...], to find $y(x) = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x} - \frac{11}{9}e^{-x} + \frac{1}{3}xe^{-x}$. □

Problem 2.

(a) Assume that the angle $\theta(t)$ of a swinging pendulum is described by $\theta'' + 4\theta = 0$. Suppose $\theta(0) = \frac{3}{10}$ and $\theta'(0) = -\frac{4}{5}$. What is the amplitude of the resulting periodic oscillations?

(b) For which values of the damping constant $c > 0$ is the system $y'' + cy' + 5y = 0$ underdamped?

(c) For which value of the external frequency ω does the system $y'' + 4y = 3\cos(\omega x)$ exhibit resonance?

(d) A forced mechanical oscillator is described by $x'' + 2x' + x = 25\cos(2t)$. What is the amplitude of the resulting steady periodic oscillations?

Solution.

(a) The characteristic equation has roots $\pm 2i$. Hence, $\theta(t) = A\cos(2t) + B\sin(2t)$.

$\theta(0) = A = \frac{3}{10}$. $\theta'(0) = 2B = -\frac{4}{5}$. Hence, $\theta(t) = \frac{3}{10}\cos(2t) - \frac{2}{5}\sin(2t) = r\cos(2t - \alpha)$ where $r(\cos\alpha, \sin\alpha) = (A, B)$.

In particular, the amplitude is $\sqrt{A^2 + B^2} = \sqrt{\frac{9}{100} + \frac{16}{100}} = \frac{1}{2}$.

(b) The characteristic equation $r^2 + cr + 5$ has roots $\frac{-c \pm \sqrt{c^2 - 20}}{2}$. The system is underdamped if the solutions involve oscillations, which happens if and only if the discriminant $\Delta = c^2 - 20$ is negative. $c^2 - 20 < 0$ if $c < \sqrt{20}$. So, the system is underdamped for $c \in (0, 2\sqrt{5})$.

(c) The natural frequency is 2 ($\pm 2i$ are the roots of the characteristic equation). Hence, there will be resonance if $\omega = 2$.

(d) The "old" roots are -1, -1. The "new" roots are $\pm 2i$. Since they don't overlap, x_{sp} has the form $x_{\text{sp}} = A\cos(2t) + B\sin(2t)$.

We plug into the DE to find $x_{\text{sp}}'' + 2x_{\text{sp}}' + x_{\text{sp}} = (-4A + 4B + A)\cos(2t) + (-4B - 4A + B)\sin(2t) \stackrel{!}{=} 25\cos(2t)$.

Solving $-3A + 4B = 25$ and $-3B - 4A = 0$, we get $B = -\frac{4}{3}A$, $\left(-3 - \frac{16}{3}\right)A = -\frac{25}{3}A = 25$. So, $A = -3$ and $B = 4$.

$x_{\text{sp}} = -3\cos(2t) + 4\sin(2t)$. In particular, the amplitude is $\sqrt{(-3)^2 + 4^2} = 5$. □

Problem 3. The motion of a certain mass on a spring is described by $x'' + x' + \frac{x}{2} = 5\sin(\omega t)$.

(a) Assume first that $\omega = 1$. Find the position function $x(t)$ if $x(0) = 2$ and $x'(0) = 0$.

Armin Straub
astraub@illinois.edu

(b) Suppose the frequency of the external force is changed so that $x'' + x' + \frac{x}{2} = 5\sin(\omega t)$. Does practical resonance occur for some value of ω? If so, for what ω?

Solution.

(a) "Old" roots $\frac{-2 \pm \sqrt{4-8}}{4} = -\frac{1}{2} \pm \frac{1}{2}i$. "New" roots $\pm i\omega$. Since there is no overlap, x_{sp} has the form $x_{\text{sp}} = A_1\cos(t) + A_2\sin(t)$. Plugging into DE, we find $A_1 = -4$, $A_2 = -2$.

Hence, the general solution is $x(t) = -4\cos(t) - 2\sin(t) + e^{-t/2}(c_1\cos(t/2) + c_2\sin(t/2))$.

Using $x(0) = -4 + c_1 = 2$, we find $c_1 = 6$. Then $x'(t) = 4\sin(t) - 2\cos(t) - \frac{1}{2}e^{-t/2}(c_1\cos(t/2) + c_2\sin(t/2)) + e^{-t/2}\left(-2\sin(t/2) + \frac{c_2}{2}\cos(t/2)\right)$. Using $x'(0) = -2 - \frac{c_1}{2} + \frac{c_2}{2} = 0$, we also find $c_2 = 10$.

In summary, $x(t) = -4\cos(t) - 2\sin(t) + e^{-t/2}(6\cos(t/2) + 10\sin(t/2))$.

(b) We proceed as in the first part, but now x_{sp} is of the form $x_{\text{sp}} = A_1\cos(\omega t) + A_2\sin(\omega t)$. Plugging into the DE, we find: first $A_2 = \frac{2\omega^2 - 1}{2\omega}A_1$, then $A_1 = -10\frac{2\omega}{(2\omega)^2 + (2\omega^2 - 1)^2} = -10\frac{2\omega}{4\omega^4 + 1}$ and so $A_2 = -10\frac{2\omega^2 - 1}{4\omega^4 + 1}$.

The amplitude is $A(\omega) = \sqrt{A_1^2 + A_2^2} = \frac{10}{\sqrt{1 + 4\omega^4}}$.

Practical resonance occurs if $A(\omega)$ has a maximum at some $\omega > 0$. To investigate, we compute $A'(\omega) = -5\frac{16\omega^3}{(1 + 4\omega^4)^{3/2}}$. We see that $A'(\omega) = 0$ only for $\omega = 0$. Hence, there is no practical resonance here. □

Problem 4. Find the general solution of $y'' - 4y' + 4y = 3e^{2x}$.

Solution. The characteristic equation for the homogeneous DE has roots 2, 2 ("old" roots). The right-hand side solves a DE whose characteristic equation has roots 2 ("new" roots). Hence, there is a particular solution of the form $y_p = Ax^2e^{2x}$.

To find A, we plug into the differential equation using $y_p' = 2A(x + x^2)e^{2x}$ and $y_p'' = 2A(1 + 4x + 2x^2)e^{2x}$:
$y_p'' - 4y_p' + 4y_p = [2A(1 + 4x + 2x^2) - 8A(x + x^2) + 4Ax^2]e^{2x} = 2Ae^{2x} \overset{!}{=} 3e^{2x}$. Hence, $A = \frac{3}{2}$.

The general solution is $\left(c_1 + c_2x + \frac{3}{2}x^2\right)e^{2x}$. □

Problem 5.

(a) Consider the differential equation $x^2y'' - 4xy' + 6y = 0$. Find all solutions of the form $y(x) = x^r$.

(b) Show that the solutions you found are independent.

(c) Note that the Wronskian of your solutions is zero for $x = 0$. Why does this not contradict the independence?

(d) Find the general solution of $x^2y'' - 4xy' + 6y = x^3$.

Solution.

(a) Plugging $y = x^r$ into the DE and assuming $r \geqslant 2$, we get $x^2r(r-1)x^{r-2} - 4xrx^{r-1} + 6x^r = [r(r-1) - 4r + 6]x^r = 0$. $r(r-1) - 4r + 6 = (r-2)(r-3)$. Hence, we have found the solutions x^2 and x^3. Since this is a second-order equation and our solutions are independent (as we will certify next), there can be no other solutions (and we can safely ignore the case $r < 2$).

(b) The Wronskian of $y_1 = x^2$, $y_2 = x^3$ is $W(x) = \det\begin{pmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{pmatrix} = x^4 \neq 0$ for $x \neq 0$.

(c) Before using the Wronskian theorem, we have to put the DE into the form $y'' - 4x^{-1}y' + 6x^{-2}y = 0$. The coefficients are not defined, and hence not continuous, for $x = 0$. We therefore cannot apply the Wronskian criterion at $x = 0$.

Armin Straub
astraub@illinois.edu

(d) Put DE in the form $y'' - 4x^{-1}y' + 6x^{-2}y = x$. The method of variation of constants shows that a particular solution is given by

$$y_p = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,\mathrm{d}x + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\,\mathrm{d}x = -x^2\int 1\,\mathrm{d}x + x^3\int \frac{1}{x}\,\mathrm{d}x = -x^3 + x^3\ln|x|.$$

Hence, the general solution is $c_1x^2 + (c_2 + \ln|x|)x^3$. ☐

Problem 6. Solve $\boldsymbol{x}' = \begin{pmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{pmatrix}\boldsymbol{x}$, $\boldsymbol{x}(0) = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$.

Solution. The characteristic polynomial is

$$\det\begin{pmatrix} 3-\lambda & -2 & 0 \\ -1 & 3-\lambda & -2 \\ 0 & -1 & 3-\lambda \end{pmatrix} = (3-\lambda)\det\begin{pmatrix} 3-\lambda & -2 \\ -1 & 3-\lambda \end{pmatrix} + 2\det\begin{pmatrix} -1 & -2 \\ 0 & 3-\lambda \end{pmatrix} = (3-\lambda)^3 - 2(3-\lambda) - 2(3-\lambda)$$
$$= (3-\lambda)[(3-\lambda)^2 - 4],$$

which has roots $\lambda = 3, 3 \pm 2 = 1, 3, 5$. These are the eigenvalues.

$\boldsymbol{\lambda = 1.}$ $\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}\boldsymbol{v} = 0$. We find $\boldsymbol{v} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

$\boldsymbol{\lambda = 3.}$ $\begin{pmatrix} 0 & -2 & 0 \\ -1 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix}\boldsymbol{v} = 0$. We find $\boldsymbol{v} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

$\boldsymbol{\lambda = 5.}$ $\begin{pmatrix} -2 & -2 & 0 \\ -1 & -2 & -2 \\ 0 & -1 & -2 \end{pmatrix}\boldsymbol{v} = 0$. We find $\boldsymbol{v} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

Consequently, the general solution is $\boldsymbol{x}(t) = c_1\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}e^t + c_2\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}e^{3t} + c_3\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}e^{5t}$.

$\boldsymbol{x}(0) = c_1\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + c_3\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}\boldsymbol{c} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$.

We eliminate: $\left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 0 & -2 & 2 \\ 1 & -1 & 1 & 6 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & -2 & -4 & 2 \\ 0 & -2 & 0 & 6 \end{array}\right)$. Hence, $c_2 = -3$, $c_3 = 1$, $c_1 = 2$.

The IVP is solved by $2\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}e^t - 3\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}e^{3t} + \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}e^{5t}$. ☐

Armin Straub
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