Net ID: **Name:**

Midterm #2

MATH 286 — Differential Equations Plus

Thursday, March 13

- No notes, personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.

Good luck!

Problem 1. (5 points) Consider a homogeneous linear differential equation with constant real coefficients which has order 6. Suppose $y(x) = x^2 e^{2x}\cos(x)$ is a solution. Write down the general solution.

Solution. The general solution is $(c_1 + c_2 x + c_3 x^2)e^{2x}\cos(x) + (c_4 + c_5 x + c_6 x^2)e^{2x}\sin(x)$. □

Problem 2. (20 points) Find the general solution of $\boldsymbol{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\boldsymbol{x}$.

Solution. The characteristic polynomial $(1-\lambda)^2 - 4$ has roots $\lambda = -1, 3$.

For $\lambda = 3$, solving $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we find the eigenvector $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda = -1$, solving $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we find the eigenvector $\boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Hence, the general solution is $\boldsymbol{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$. □

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Problem 3. (10 points) The position $x(t)$ of a certain mass on a spring is described by $x'' + cx' + 5x = F\sin(\omega t)$.

(a) Assume first that there is no external force, i.e. $F = 0$. For which values of c is the system overdamped?

(b) Now, $F \neq 0$ and the system is undamped, i.e. $c = 0$. For which values of ω, if any, does resonance occur?

Solution.

(a) The discriminant of the characteristic equation is $c^2 - 20$. Hence the system is overdamped if $c^2 - 20 > 0$, that is $c > \sqrt{20} = 2\sqrt{5}$.

(b) The natural frequency is $\sqrt{5}$. Resonance therefore occurs if $\omega = \sqrt{5}$. □

Problem 4. (20 points) Find the general solution of the differential equation $y^{(3)} - y = e^x + 7$.

Solution. Let us first solve the homogeneous equation $y''' - y = 0$. Its characteristic polynomial $r^3 - 1 = (r - 1)(r^2 + r + 1)$ has roots $r = 1$ and $r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

There is a particular solution of the form $y_p = Axe^x + B$.

$y_p' = A(x+1)e^x,\ y_p'' = A(x+2)e^x,\ y_p''' = A(x+3)e^x$

Plugging into the DE, we get $y_p''' - y_p = 3Ae^x - B \overset{!}{=} e^x + 7$. Consequently, $A = \frac{1}{3}$, $B = -7$.

Hence, the general solution is $-7 + (c_1 + \frac{1}{3}x)e^x + c_2 e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$. □

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Problem 5. (20 points) Consider, for $x>0$, the second-order differential equation

$$y'' - \left(1 + \frac{2}{x}\right) y' + \left(\frac{1}{x} + \frac{2}{x^2}\right) y = 0.$$

(a) Show that the the functions $y_1(x) = x$ and $y_2(x) = x\,e^x$ are solutions to this differential equation.

(b) Using the Wronskian, show that y_1 and y_2 are linearly independent solutions to the above differential equation.

(c) Find, for $x>0$, the general solution to the second-order differential equation

$$y'' - \left(1 + \frac{2}{x}\right) y' + \left(\frac{1}{x} + \frac{2}{x^2}\right) y = 2x.$$

Solution.

(a) We have

$$y_1'' - \left(1 + \frac{2}{x}\right) y_1' + \left(\frac{1}{x} + \frac{2}{x^2}\right) y_1 = 0 - \left(1 + \frac{2}{x}\right) + \left(\frac{1}{x} + \frac{2}{x^2}\right) x = 0,$$

and

$$y_2'' - \left(1 + \frac{2}{x}\right) y_2' + \left(\frac{1}{x} + \frac{2}{x^2}\right) y_2 = x\,e^x + 2e^x - \left(1 + \frac{2}{x}\right)(x\,e^x + e^x) + \left(\frac{1}{x} + \frac{2}{x^2}\right)(x\,e^x) = 0.$$

(b) The Wronskian of y_1 and y_2 is given by

$$W(x) = \det\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = \det\begin{bmatrix} x & x\,e^x \\ 1 & x\,e^x + e^x \end{bmatrix} = x^2\,e^x.$$

Since $W(x) \neq 0$ on the domain of definition for the differential equation, the Wronskian theorem implies that y_1 and y_2 are linearly independent.

(c) The general solution is given by

$$y(x) = c_1\,y_1(x) + c_2\,y_2(x) + y_p(x) = c_1\,x + c_2\,x\,e^x + y_p(x),$$

where y_p is any particular solution to the above non-homogeneous equation. To find such a y_p, we will use the variation of parameters formula:

$$y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x) = u_1(x)\,x + u_2(x)\,x\,e^x,$$

where

$$\begin{aligned} u_1(x) &= -\int \frac{2x\,y_2(x)}{W(x)}\,\mathrm{d}x = -\int 2\mathrm{d}x = -2x, \\ u_2(x) &= \int \frac{2x\,y_1(x)}{W(x)}\,\mathrm{d}x = \int 2\,e^{-x}\mathrm{d}x = -2\,e^{-x}. \end{aligned}$$

So,

$$y(x) = c_1\,x + c_2\,x\,e^x - 2x^2 - 2x = d_1 x + d_2 x\,e^x - 2x^2.$$

□

Problem 6. (20 points) The motion of a certain mass on a spring is described by $x'' + 2x' + 2x = 5\sin(t)$.

(a) What is the amplitude of the resulting steady periodic oscillations?

(b) Assume that the mass is initially at rest (i.e. $x(0) = 0$, $x'(0) = 0$) and find the position function $x(t)$.

Solution.

(a) The characteristic polynomial of the associated homogeneous DE has roots $\frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$.

Hence, x_{sp} is the form $x_{\text{sp}} = A_1\cos(t) + A_2\sin(t)$.

We compute $x_{\text{sp}}' = -A_1\sin(t) + A_2\cos(t)$ and $x_{\text{sp}}'' = -A_1\cos(t) - A_2\sin(t)$.

Plugging into the DE gives $x_{\text{sp}}'' + 2x_{\text{sp}}' + 2x_{\text{sp}} = (A_1 + 2A_2)\cos(t) + (A_2 - 2A_1)\sin(t) \overset{!}{=} 5\sin(t)$. Consequently, $A_1 + 2A_2 = 0$ and $A_2 - 2A_1 = 5$, resulting in $A_1 = -2$, $A_2 = 1$.

Thus, $x_{\text{sp}} = -2\cos(t) + \sin(t)$. The amplitude is $\sqrt{(-2)^2 + 1^2} = \sqrt{5}$.

(b) From first part, we know that $x(t) = -2\cos(t) + \sin(t) + e^{-t}(c_1\cos(t) + c_2\sin(t))$.

Using $x(0) = -2 + c_1 = 0$ we find $c_1 = 2$.

$x'(t) = 2\sin(t) + \cos(t) - e^{-t}(2\cos(t) + c_2\sin(t)) + e^{-t}(-2\sin(t) + c_2\cos(t))$. Hence, $x'(0) = 1 - 2 + c_2 = 0$ results in $c_2 = 1$.

In conclusion, $x(t) = -2\cos(t) + \sin(t) + e^{-t}(2\cos(t) + \sin(t))$. □

Problem 7. (5 points) Let y_p be any solution to the inhomogeneous linear differential equation $y'' + xy = e^x$. Find a homogeneous linear differential equation which y_p solves. *Hint:* Do not attempt to solve the DE.

Solution. Apply $D = \frac{\mathrm{d}}{\mathrm{d}x}$ to both sides of the differential equation to get $y''' + xy' + y = e^x$. Subtracting the two differential equations, we get the homogeneous linear DE $y''' - y'' + xy' + (1-x)y = 0$.

[Note that this is the same as applying $D - 1$ to both sides of the DE, in analogy with our approach for solving inhomogeneous linear DEs with constant coefficients.] □