Review: Examples of differential equations we can solve

Let's start with one of the simplest (and most fundamental) differential equation (DE). It is **first-order** (only a first derivative) and **linear** (with constant coefficients).

Example 1. Solve $y' = 3y$.

   Solution. $y(x) = Ce^{3x}$

   Check. Indeed, if $y(x) = Ce^{3x}$, then $y'(x) = 3Ce^{3x} = 3y(x)$.

   Comment. Recall we can always easily check whether a function solves a differential equation. This means that (although you might be unfamiliar with the techniques for solving) you can use computer algebra systems like Sage to solve differential equations without trust issues.

To describe a unique solution, additional constraints need to be imposed.

Example 2. Solve the **initial value problem** (IVP) $y' = 3y$, $y(0) = 5$.

   Solution. This has the unique solution $y(x) = 5e^{3x}$.

The following is a **non-linear** differential equation. In general, such equations are much more complicated than linear ones. We can solve this particular one because it is **separable**.

Example 3. Solve $y' = xy^2$.

   Solution. This DE is separable: $\frac{1}{y^2}\mathrm{d}y = x\,\mathrm{d}x$. Integrating, we find $-\frac{1}{y} = \frac{1}{2}x^2 + C$.

   Hence, $y = -\frac{1}{\frac{1}{2}x^2 + C} = \frac{2}{D - x^2}$.

   [Here, $D = -2C$ but that relationship doesn't matter; it only matters that the solution has a free parameter.]

   Note. Note that we did not find the solution $y = 0$ (lost when dividing by y^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above).

   Check. Compute y' and verify that the DE is indeed satisfied.

Excursion: Euler's identity

Theorem 4. (Euler's identity) $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

   Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.

   [Check that by computing the derivatives and verifying the initial condition! As we did in class.]   □

   On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Example 5. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1-\cos(2x)}{2}$ (and infinitely many others you have never heard of!) come from?

Short answer: they all come from the simple exponential law $e^{x+y} = e^x e^y$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$
\begin{aligned}
e^{2ix} &= \cos(2x) + i\sin(2x) \\
e^{ix}e^{ix} &= [\cos(x) + i\sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x).
\end{aligned}
$$

Comparing imaginary parts (the "stuff with an i"), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.

Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1-\cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?

Or, use $e^{i(x+y)} = e^{ix}e^{iy}$ to derive $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x+y) = ...$ (that's what we actually did in class).

Realize that the complex number $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ corresponds to the point $(\cos(\theta), \sin(\theta))$.

These are precisely the points on the unit circle!

Recall that a point (x, y) can be represented using **polar coordinates** (r, θ), where r is the distance to the origin and θ is the angle with the x-axis.

Then, $x = r\cos\theta$ and $y = r\sin\theta$.

Every complex number z can be written in **polar form** as $z = re^{i\theta}$, with $r = |z|$.

Why? By comparing with the usual polar coordinates ($x = r\cos\theta$ and $y = r\sin\theta$), we can write

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}.$$

In the final step, we used Euler's identity.