Review: Linear DEs

A linear DE of order n is of the form $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = f(x)$.

- In terms of $D = \frac{\mathrm{d}}{\mathrm{d}x}$, the DE becomes: $Ly = f(x)$ with $L = D^n + p_{n-1}(x) D^{n-1} + \ldots + p_1(x) D + p_0(x)$.
  Comment. L is called a (linear) differential operator.
- The inclusion of the $f(x)$ term makes $Ly = f(x)$ an **inhomogeneous** linear DE.
- $Ly = 0$ is the corresponding **homogeneous** DE.
  - If y_1 and y_2 are solutions to the homogeneous DE, then so is any linear combination $C_1 y_1 + C_2 y_2$.
  - **(general solution of the homogeneous DE)** There are n solutions $y_1, y_2, \ldots, y_n$, such that every solution is of the form $C_1 y_1 + \ldots + C_n y_n$. [These n solutions necessarily are **independent**.]
- To find the general solution of the inhomogeneous DE, we only need to find a single solution y_p (called a **particular solution**). Then the general solution is $y_p + y_h$, where y_h is the general solution of the homogeneous DE.

Linear first-order DEs

The following DE is linear and first-order (but not with constant coefficients).

Example 15. Solve $y' = x^2 y$.

Solution. This DE is separable as well: $\frac{1}{y}\mathrm{d}y = x^2\,\mathrm{d}x$ (note that we just lost the solution $y = 0$).

Integrating gives $\ln|y| = \frac{1}{3}x^3 + C$, so that $|y| = e^{\frac{1}{3}x^2 + C}$. Since the RHS is never zero, $y = \pm e^C e^{\frac{1}{3}x^3} = D e^{\frac{1}{3}x^3}$ (with $D = \pm e^C$). Note that $D = 0$ corresponds to the singular solution $y = 0$.

In summary, the general solution is $y = D e^{\frac{1}{3}x^3}$ (with D any real number).

Check. Compute y' and verify that the DE is indeed satisfied.

As in the previous example, we can immediately solve any **homogeneous** linear first-order DE:

Example 16. Solve $y' = a(x) y$.

Solution. Proceeding as in the previous example, we find $y(x) = D e^{\int a(x)\mathrm{d}x}$.

Check. Compute y' and verify that the DE is indeed satisfied.

Recall that, to find the general solution of the inhomogeneous DE $y' = a(x) y + f(x)$, we only need to find a particular solution y_p.

Then the general solution is $y_p + y_h$, where y_h is the general solution of the homogeneous DE $y' = a(x) y$.

Theorem 17. $y' = a(x) y + f(x)$ has the particular solution

$$y_p(x) = y_h(x) \int \frac{f(x)}{y_h(x)} \mathrm{d}x,$$

where $y_h(x) = e^{\int a(x)\mathrm{d}x}$ is any solution to the homogeneous equation $y' = a(x) y$.

Proof. $y_p'(x) = y_h'(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x + y_h(x)\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\int \frac{f(x)}{y_h(x)}\mathrm{d}x}_{\frac{f(x)}{y_h(x)}} = a(x)y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x + f(x) = a(x)y + f(x)$ □

Comment. Note that the formula for $y_p(x)$ gives the general solution if we let $\int \frac{f(x)}{y_h(x)}\mathrm{d}x$ be the general antiderivative. (Think about the effect of the constant of integration!)

Recall. The formula for $y_p(x)$ can be found using **variation of constants** (sometimes called variation of parameters): that is, we look for solutions of the form $y(x) = c(x)y_h(x)$.

If we plug $y(x) = c(x)y_h(x)$ into the DE, we find $c'y_h + cy_h' = acy_h + f$. Since $y_h' = ay_h$, this simplifies to $c'y_h = f$ or, equivalently, $c' = \frac{f}{y_h}$. Hence, $c(x) = \int \frac{f(x)}{y_h(x)}\mathrm{d}x$, which is the formula in the theorem.

Example 18. Solve $x^2y' = 1 - xy + 2x$, $y(1) = 3$.

Solution. Write as $\frac{\mathrm{d}y}{\mathrm{d}x} = a(x)y + f(x)$ with $a(x) = -\frac{1}{x}$ and $f(x) = \frac{1}{x^2} + \frac{2}{x}$.

$y_h(x) = e^{\int a(x)\mathrm{d}x} = e^{-\ln x} = \frac{1}{x}$. (Why can we write $\ln x$ instead of $\ln|x|$?!) Hence:

$$y_p(x) = y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x = \frac{1}{x}\int \left(\frac{1}{x} + 2\right)\mathrm{d}x = \frac{\ln x + 2x + C}{x}$$

Using $y(1) = 3$, we find $C = 1$. In summary, the solution is $y = \frac{\ln(x) + 2x + 1}{x}$.

Comment. Observe how the general solution (with parameter C) is indeed obtained from any particular solution (say, $\frac{\ln x + 2x}{x}$) plus the general solution to the homogeneous equation, which is $\frac{C}{x}$.

Homogeneous linear DEs with constant coefficients

Example 19. Find the general solution of $y'' - y' - 2y = 0$.

Solution. We recall from *Differential Equations I* that e^{rx} solves this DE for the right choice of r.

Plugging e^{rx} into the DE, we get $r^2e^{rx} - re^{rx} - 2e^{rx} = 0$.

Equivalently, $r^2 - r - 2 = 0$. This is called the **characteristic equation**. Its solutions are $r = 2, -1$.

This means we found the two solutions $y_1 = e^{2x}$, $y_2 = e^{-x}$.

Since this a homogeneous linear DE, the general solution is $y = C_1e^{2x} + C_2e^{-x}$.

Next time. A useful way to look at these kinds of differential equations through differential operators.

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