Example 29. (review) Find the general solution of $y''' - y'' - 5y' - 3y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - D^2 - 5D - 3 = (D-3)(D+1)^2$ has roots $3, -1, -1$.
Hence, the general solution is $y(x) = C_1 e^{3x} + (C_2 + C_3 x)e^{-x}$.

Example 30. Find the general solution of $y'' + 4y = 12x$.

Solution. Here, $p(D) = D^2 + 4$, which has roots $\pm 2i$.

Hence, the general solution is $y(x) = y_p(x) + C_1 \cos(2x) + C_2 \sin(2x)$. It remains to find a particular solution y_p.

Noting that $D^2 \cdot (12x) = 0$, we apply D^2 to both sides of the DE.

We get $D^2(D^2 + 4) \cdot y = 0$, which is a homogeneous linear DE! Its general solution is $C_1 + C_2 x + C_3 \cos(2x) + C_4 \sin(2x)$. In particular, y_p is of this form for some choice of $C_1, \ldots, C_4$.

It simplifies our life to note that there has to be a particular solution of the simpler form $y_p = C_1 + C_2 x$.

[Why?! Because we know that $C_3 \cos(2x) + C_4 \sin(2x)$ can be added to any particular solution.]

It only remains to find appropriate values C_1, C_2 such that $y_p'' + 4y_p = 12x$. Since $y_p'' + 4y_p = 4C_1 + 4C_2 x$, comparing coefficients yields $4C_1 = 0$ and $4C_2 = 12$, so that $C_1 = 0$ and $C_2 = 3$. In other words, $y_p = 3x$.

Therefore, the general solution to the original DE is $y(x) = 3x + C_1 \cos(2x) + C_2 \sin(2x)$.

Example 31. Find the general solution of $y'' + 4y' + 4y = e^x$.

Solution. This is $p(D)y = e^x$ with $p(D) = D^2 + 4D + 4 = (D+2)^2$.

Hence, the general solution is $y(x) = y_p(x) + (C_1 + C_2 x)e^{-2x}$. It remains to find a particular solution y_p.

Note that $(D-1)e^x = 0$. Hence, we apply $(D-1)$ to the DE to get $(D-1)(D+2)^2 y = 0$.

This homogeneous linear DE has general solution $(C_1 + C_2 x)e^{-2x} + C_3 e^x$. We conclude that the original DE must have a particular solution of the form $y_p = C_3 e^x$.

To determine the value of C_3, we plug into the original DE: $y_p'' + 4y_p' + 4y_p = 9C_3 e^x \overset{!}{=} e^x$. Hence, $C_3 = 1/9$.

In conclusion, the general solution is $y(x) = (C_1 + C_2 x)e^{-2x} + \frac{1}{9} e^x$.

We found a recipe for solving nonhomogeneous linear DEs with constant coefficients.

Our approach works for $p(D)y = f(x)$ whenever the right-hand side $f(x)$ is the solution of some homogeneous linear DE with constant coefficients: $q(D) f(x) = 0$

Theorem 32. To find a particular solution y_p to an inhomogeneous linear DE with constant coefficients $p(D)y = f(x)$:

- Find $q(D)$ so that $q(D) f(x) = 0$. [This does not work for all $f(x)$.]
- Let $r_1, \ldots, r_n$ be the ("old") roots of the polynomial $p(D)$.
  Let $s_1, \ldots, s_m$ be the ("new") roots of the polynomial $q(D)$.
- It follows that y_p solves $q(D)\, p(D) y = 0$.
  The characteristic polynomial of this DE has roots $r_1, \ldots, r_n, s_1, \ldots, s_m$.
  Let $v_1, \ldots, v_m$ be the "new" solutions (i.e. not solutions of the "old" $p(D)y = 0$).
  By plugging into $p(D) y_p = f(x)$, we find (unique) C_i so that $y_p = C_1 v_1 + \ldots + C_m v_m$.

For which $f(x)$ does this work? By Theorem 25, we know exactly which $f(x)$ are solutions to homogeneous linear DEs with constant coefficients: these are linear combinations of exponentials $x^j e^{rx}$ (which includes $x^j e^{ax} \cos(bx)$ and $x^j e^{ax} \sin(bx)$).

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Example 33. Find the general solution of $y''+4y'+4y=7e^{-2x}$.

Solution. The "old" roots are $-2,-2$. The "new" roots are -2. Hence, there has to be a particular solution of the form $y_p=Cx^2e^{-2x}$. To find the value of C, we plug into the DE.

$y_p'=C(-2x^2+2x)e^{-2x}$

$y_p''=C(4x^2-8x+2)e^{-2x}$

$y_p''+4y_p'+4y_p=2Ce^{-2x}\overset{!}{=}7e^{-2x}$

It follows that $C=7/2$, so that $y_p=\frac{7}{2}x^2e^{-2x}$. The general solution is $y(x)=\left(C_1+C_2x+\frac{7}{2}x^2\right)e^{-2x}$.

Example 34. Find a particular solution of $y''+4y'+4y=x\cos(x)$.

Solution. The "old" roots are $-2,-2$. The "new" roots are $\pm i,\pm i$. Hence, there has to be a particular solution of the form $y_p=(C_1+C_2x)\cos(x)+(C_3+C_4x)\sin(x)$. To find the value of the C_j's, we plug into the DE.

$y_p'=(C_2+C_3+C_4x)\cos(x)+(C_4-C_1-C_2x)\sin(x)$

$y_p''=(2C_4-C_1-C_2x)\cos(x)+(-2C_2-C_3-C_4x)\sin(x)$

$y_p''+4y_p'+4y_p=(3C_1+4C_2+4C_3+2C_4+(3C_2+4C_4)x)\cos(x)$

$+(-4C_1-2C_2+3C_3+4C_4+(-4C_2+3C_4)x)\sin(x)\overset{!}{=}x\cos(x)$.

Equating the coefficients of $\cos(x)$, $x\cos(x)$, $\sin(x)$, $x\sin(x)$, we get the equations $3C_1+4C_2+4C_3+2C_4=0$, $3C_2+4C_4=1$, $-4C_1-2C_2+3C_3+4C_4=0$, $-4C_2+3C_4=0$.

Solving, we find $C_1=-\frac{4}{125}$, $C_2=\frac{3}{25}$, $C_3=-\frac{22}{125}$, $C_4=\frac{4}{25}$. [Make sure you know how to do this tedious step.]

Hence, $y_p=\left(-\frac{4}{125}+\frac{3}{25}x\right)\cos(x)+\left(-\frac{22}{125}+\frac{4}{25}x\right)\sin(x)$.

Example 35. (extra) Find a particular solution of $y''+4y'+4y=5e^{-2x}-3x\cos(x)$.

Solution. Instead of starting all over, recall that we already found $y_\triangle$ in Example 33 such that $Ly_\triangle=7e^{-2x}$ (here, we write $L=p(D)$). Also, from Example 34 we have $y_\diamond$ such that $Ly_\diamond=x\cos(x)$.

By linearity, it follows that $L\left(\frac{5}{7}y_\triangle-3y_\diamond\right)=\frac{5}{7}Ly_\triangle-3Ly_\diamond=5e^{-2x}-3x\cos(x)$.

Hence, $y_p=\frac{5}{7}y_\triangle-3y_\diamond=\frac{5}{2}x^2e^{-2x}-3\left[\left(-\frac{4}{125}+\frac{3}{25}x\right)\cos(x)+\left(-\frac{22}{125}+\frac{4}{25}x\right)\sin(x)\right]$.

Example 36. (extra) Find a particular solution of $y''+4y'+4y=4e^{3x}\sin(2x)-x\sin(x)$.

Solution. The "old" roots are $-2,-2$. The "new" roots are $3\pm 2i,\pm i,\pm i$.

Hence, there has to be a particular solution of the form

$y_p=C_1e^{3x}\cos(2x)+C_2e^{3x}\sin(2x)+(C_3+C_4x)\cos(x)+(C_5+C_6x)\sin(x)$.

To find the values of $C_1,\ldots,C_6$, we plug into the DE. But this final step is so boring that we stop here.

Computers (currently?) cannot afford to be as selective; mine obediently calculated:

$y_p=-\frac{4}{841}e^{3x}(20\cos(2x)-21\sin(2x))+\frac{1}{125}((-22+20x)\cos(x)+(4-15x)\sin(x))$

Example 37. We have been factoring differential operators like $D^2+4D+4=(D+2)^2$.

Things become much more complicated, when the coefficients are not constant!

For instance, the linear DE $y''+4y'+4xy=0$ can be written as $Ly=0$ with $L=D^2+4D+4x$. However, in general, such operators cannot be factored (unless we allow as coefficients functions in x that we are not familiar with). [On the other hand, any ordinary polynomial can be factored over the complex numbers.]

One indication that things become much more complicated is that x and D do not commute: $xD\neq Dx$!!

Indeed, $(xD)f(x)=xf'(x)$ while $(Dx)f(x)=\frac{\mathrm{d}}{\mathrm{d}x}[xf(x)]=f(x)+xf'(x)=(1+xD)f(x)$.

This computation shows that, in fact, $Dx=xD+1$.

More next time!