Review. Linear DEs are those that can be written as $Ly = f(x)$ where L is a linear differential operator: namely,

$$L = p_n(x) D^n + p_{n-1}(x) D^{n-1} + \ldots + p_1(x) D + p_0(x). \quad (1)$$

Recall that the operators xD and Dx are not the same: instead, $Dx = xD + 1$.
We say that an operator of the form (1) is in **normal form**.

For instance. xD is in normal, whereas Dx is not in normal form. The normal form of Dx is $xD + 1$.

Example 38. Let $a = a(x)$ be some function.

(a) Write the operator Da in normal form [normal form means as in (1)].

(b) Write the operator $D^2 a$ in normal form.

Solution.

(a) $(Da) f(x) = \frac{\mathrm{d}}{\mathrm{d}x}[a(x)\, f(x)] = a'(x) f(x) + a(x) f'(x) = (a' + aD) f(x)$
Hence, $Da = aD + a'$.

(b) $(D^2 a) f(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}[a(x)\, f(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[a'(x) f(x) + a(x) f'(x)] = a''(x) f(x) + 2a'(x) f'(x) + a(x) f''(x)$
$= (a'' + 2a' D + a D^2) f(x)$
Hence, $D^2 a = a D^2 + 2a' D + a''$.

Example 39. Suppose that a and b depend on x. Expand $(D + a)(D + b)$ in normal form.

Solution. $(D + a)(D + b) = D^2 + Db + aD + ab = D^2 + (bD + b') + aD + ab = D^2 + (a + b) D + ab + b'$

Comment. Of course, if b is a constant, then $b' = 0$ and we just get the familiar expansion.

Comment. At this point, it is not surprising that, in general, $(D + a)(D + b) \neq (D + b)(D + a)$.

Example 40. Suppose we want to factor $D^2 + pD + q$ as $(D + a)(D + b)$. [p, q, a, b depend on x]

(a) Spell out equations to find a and b.

(b) Find all factorizations of D^2. [An obvious one is $D^2 = D \cdot D$ but there is others!]

Solution.

(a) Matching coefficients with $(D + a)(D + b) = D^2 + (a + b) D + ab + b'$, we find that we need

$$p = a + b, \quad q = ab + b'.$$

Equivalently, $a = p - b$ and $q = (p - b) b + b'$. The latter is a nonlinear (!) DE for b. Once solved for b, we obtain a as $a = p - b$.

(b) This is the case $p = q = 0$. The DE for b becomes $b' = b^2$.
Because it is separable (show all details!), we find that $b(x) = \frac{1}{C - x}$ or $b(x) = 0$.
Since $a = -b$, we obtain the factorizations $D^2 = \left(D - \frac{1}{C - x}\right)\left(D + \frac{1}{C - x}\right)$ and $D^2 = D \cdot D$.
Our computations show that there are no further factorizations.

Comment. Note that this example illustrates that factorization of differential operators is not unique!
For instance, $D^2 = D \cdot D$ and $D^2 = \left(D + \frac{1}{x}\right) \cdot \left(D - \frac{1}{x}\right)$ (the case $C = 0$ above).

Comment. In general, the nonlinear DE for b does not have any polynomial or rational solution (or, in fact, any solution that can be expressed in terms of functions that we are familiar with).

Armin Straub
straub@southalabama.edu

A crash course in computing determinants

Review. The **determinant** of A, written as $\det(A)$ or $|A|$, is a number with the property that:

$$\begin{aligned}\det(A)\neq 0 &\iff A \text{ is invertible}\\ &\iff A\boldsymbol{x}=\boldsymbol{b} \text{ has a (unique) solution } \boldsymbol{x} \text{ (for all } \boldsymbol{b})\\ &\iff A\boldsymbol{x}=0 \text{ is only solved by } \boldsymbol{x}=0\end{aligned}$$

Example 41. $\begin{vmatrix} a & b\\ c & d\end{vmatrix}=ad-bc$

Example 42. Compute $\begin{vmatrix} 1 & 2 & 0\\ 3 & -1 & 2\\ 2 & 0 & 1\end{vmatrix}$ by **cofactor expansion**.

Solution. We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0\\ 3 & -1 & 2\\ 2 & 0 & 1\end{vmatrix}=1\cdot\begin{vmatrix} + & \blacksquare & \blacksquare\\ \blacksquare & -1 & 2\\ \blacksquare & 0 & 1\end{vmatrix}-2\cdot\begin{vmatrix} \blacksquare & - & \blacksquare\\ 3 & \blacksquare & 2\\ 2 & \blacksquare & 1\end{vmatrix}+0\cdot\begin{vmatrix} \blacksquare & \blacksquare & +\\ 3 & -1 & \blacksquare\\ 2 & 0 & \blacksquare\end{vmatrix}$$

$$\overset{\text{i.e.}}{=}1\cdot\begin{vmatrix} -1 & 2\\ 0 & 1\end{vmatrix}-2\cdot\begin{vmatrix} 3 & 2\\ 2 & 1\end{vmatrix}+0\cdot\begin{vmatrix} 3 & -1\\ 2 & 0\end{vmatrix}=1\cdot(-1)-2\cdot(-1)+0=1$$

Each term in the cofactor expansion is ± 1 times an entry times a smaller determinant (row and column of entry deleted).

The ± 1 is assigned to each entry according to $\begin{bmatrix} + & - & + & \cdots\\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots\end{bmatrix}$.

Solution. We expand by the second column:

$$\begin{vmatrix} 1 & 2 & 0\\ 3 & -1 & 2\\ 2 & 0 & 1\end{vmatrix}=-2\cdot\begin{vmatrix} \blacksquare & - & \blacksquare\\ 3 & \blacksquare & 2\\ 2 & \blacksquare & 1\end{vmatrix}+(-1)\cdot\begin{vmatrix} 1 & \blacksquare & 0\\ \blacksquare & + & \blacksquare\\ 2 & \blacksquare & 1\end{vmatrix}-0\cdot\begin{vmatrix} 1 & \blacksquare & 0\\ 3 & \blacksquare & 2\\ \blacksquare & - & \blacksquare\end{vmatrix}$$

$$=-2\cdot(-1)+(-1)\cdot 1-0=1$$

Example 43. Compute $\begin{vmatrix} 1 & 0 & 3 & 4\\ 0 & 2 & 1 & 5\\ 0 & 0 & 2 & 1\\ 2 & 0 & 8 & 5\end{vmatrix}$.

Solution. We can expand by the second column:

$$\begin{vmatrix} 1 & 0 & 3 & 4\\ 0 & 2 & 1 & 5\\ 0 & 0 & 2 & 1\\ 2 & 0 & 8 & 5\end{vmatrix}=-0\begin{vmatrix} 0 & 1 & 5\\ 0 & 2 & 1\\ 2 & 8 & 5\end{vmatrix}+2\begin{vmatrix} 1 & 3 & 4\\ 0 & 2 & 1\\ 2 & 8 & 5\end{vmatrix}-0\begin{vmatrix} 1 & 3 & 4\\ 0 & 1 & 5\\ 2 & 8 & 5\end{vmatrix}+0\begin{vmatrix} 1 & 3 & 4\\ 0 & 1 & 5\\ 0 & 2 & 1\end{vmatrix}$$

[Of course, you don't have to spell out the 3×3 matrices that get multiplied with 0.]

We can compute the remaining 3×3 matrix in any way we prefer. One option is to expand by the first column:

$$2\begin{vmatrix} 1 & 3 & 4\\ 0 & 2 & 1\\ 2 & 8 & 5\end{vmatrix}=2\left(+1\begin{vmatrix} 2 & 1\\ 8 & 5\end{vmatrix}+2\begin{vmatrix} 3 & 4\\ 2 & 1\end{vmatrix}\right)=2(1\cdot 2+2\cdot(-5))=-16$$

Comment. For cofactor expansion, choosing to expand by the second column is the best choice because this column has more zeros than any other column or row.