Example 59. Write $a_{n+3}-4a_{n+2}+a_{n+1}+6a_n=0$ as a system of (first-order) recurrences.

Solution. Write $b_n=a_{n+1}$ and $c_n=a_{n+2}$.

Then, $a_{n+3}-4a_{n+2}+a_{n+1}+6a_n=0$ translates into the first-order system $\begin{cases} a_{n+1}=b_n \\ b_{n+1}=c_n \\ c_{n+1}=-6a_n-b_n+4c_n \end{cases}$.

Let $\boldsymbol{a}_n=\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}$. Then, in matrix form, the RE is $\boldsymbol{a}_{n+1}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix}\boldsymbol{a}_n$.

Review.

- Consequently, $\boldsymbol{a}_n=M^n\boldsymbol{a}_0$, where M is the matrix above.
- In general, we can solve $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$ by finding the eigenvectors of M:
  An λ-eigenvector $\boldsymbol{v}$ provides the solution $\boldsymbol{a}_n=\boldsymbol{v}\lambda^n$.
- Here, because we started with a single (third-order) equation, we can avoid computing eigenvectors:
  $a_n=C_1\cdot 3^n+C_2\cdot 2^n+C_3\cdot(-1)^n$ is the general solution to the initial RE. (Why?! Do it!)

  Correspondingly, a fundamental matrix solution of the system is $\Phi_n=\begin{bmatrix} 3^n & 2^n & (-1)^n \\ 3\cdot 3^n & 2\cdot 2^n & -(-1)^n \\ 9\cdot 3^n & 4\cdot 2^n & (-1)^n \end{bmatrix}$.

  Note. This tells us that $\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$ is a 3-eigenvector, $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ a 2-eigenvector, and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ a -1-eigenvector of M.
- Since $\Phi_{n+1}=M\Phi_n$, we have $\Phi_n=M^n\Phi_0$ so that $M^n=\Phi_n\Phi_0^{-1}$. This allows us to compute that:
  $$M^n=\frac{1}{12}\begin{bmatrix} -6\cdot 3^n+12\cdot 2^n+6(-1)^n & -3\cdot 3^n+8\cdot 2^n-5(-1)^n & 3\cdot 3^n-4\cdot 2^n+(-1)^n \\ -18\cdot 3^n+24\cdot 2^n-6(-1)^n & \ldots & \ldots \\ -54\cdot 3^n+48\cdot 2^n+6(-1)^n & \ldots & \ldots \end{bmatrix}$$

(systems of REs) The unique solution to $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$, $\boldsymbol{a}_0=\boldsymbol{c}$ is $\boldsymbol{a}_n=M^n\boldsymbol{c}$.

Note that M^n is the fundamental matrix solution to $\boldsymbol{a}_{n+1}=M\boldsymbol{a}_n$ with $\boldsymbol{a}_0=I$ (the identity matrix).

Systems of differential equations

Example 60. Write the (second-order) differential equation $y''=2y'+y$ as a system of (first-order) differential equations.

Solution. Write $y_1=y$ and $y_2=y'$. Then $y''=2y'+y$ becomes $y_2'=2y_2+y_1$.

Therefore, $y''=2y'+y$ translates into the first-order system $\begin{cases} y_1'=y_2 \\ y_2'=y_1+2y_2 \end{cases}$.

In matrix form, this is $\boldsymbol{y}'=\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\boldsymbol{y}$.

Comment. Hence, we care about systems of differential equations, even if we work with just one function.

Example 61. Write the (third-order) differential equation $y'''=3y''-2y'+y$ as a system of (first-order) differential equations.

Solution. Write $y_1=y$, $y_2=y'$ and $y_3=y''$.

Then, $y'''=3y''-2y'+y$ translates into the first-order system $\begin{cases} y_1'=y_2 \\ y_2'=y_3 \\ y_3'=y_1-2y_2+3y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}'=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix}\boldsymbol{y}$.

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Example 62. Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 2y_1' - 3y_2' + 7y_2 \\ y_2'' &= 4y_1' + y_2' - 5y_1 \end{aligned} \qquad y_1(0) = 2,\ y_1'(0) = 3,\ y_2(0) = -1,\ y_2'(0) = 1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

(systems of DEs) The unique solution to $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$ is $\boldsymbol{y}(x) = e^{Mx}\boldsymbol{c}$.

Here, e^{Mx} is the fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = I$ (with I the identity matrix).

Important. We are defining the **matrix exponential** e^{Mx} as the solution to an IVP. This is equivalent to how one can define the ordinary exponential e^x as the solution to $y' = y$, $y(0) = 1$.

[In a little bit, we will also discuss how to think about the matrix exponential e^{Mx} using power series.]

(a way to compute the matrix exponential e^{Mx})

Compute a fundamental matrix solution $\Phi(x)$ of $\boldsymbol{y}' = M\boldsymbol{y}$.

Then $e^{Mx} = \Phi(x)\Phi(0)^{-1}$.

Compare this to our method of computing matrix powers M^n.

Proof. If $\Phi(x)$ is a fundamental matrix solution, then so is $\Psi(x) = \Phi(x)C$ for any constant matrix C. (Why?!)

Therefore, $\Psi(x) = \Phi(x)\Phi(0)^{-1}$ is a fundamental matrix solution with $\Psi(0) = \Phi(0)\Phi(0)^{-1} = I$.

But e^{Mx} is defined to be the unique such solution, so that $\Psi(x) = e^{Mx}$.

Observe how the next example proceeds along the same lines as Example 58.

Example 63. Let $M = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

Solution.

(a) Let us look for solutions of the form $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$ (where $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$). Note that $\boldsymbol{y}' = \lambda\boldsymbol{v}e^{\lambda x} = \lambda\boldsymbol{y}$.

Plugging into $\boldsymbol{y}' = M\boldsymbol{y}$ we find $\lambda\boldsymbol{y} = M\boldsymbol{y}$.

In other words, $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$ is a solution if and only if $\boldsymbol{v}$ is a λ-eigenvector of M.

We computed earlier that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.

Hence, the general solution is $C_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{3x} + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2x}$.

(b) The corresponding fundamental matrix solution is $\Phi = \begin{bmatrix} 2 \cdot e^{3x} & e^{-2x} \\ e^{3x} & e^{-2x} \end{bmatrix}$.

[Note that our general solution is precisely $\Phi\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.]

(c) Note that $\Phi(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} 2 \cdot e^{3x} & e^{-2x} \\ e^{3x} & e^{-2x} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot e^{3x} - e^{-2x} & -2 \cdot e^{3x} + 2e^{-2x} \\ e^{3x} - e^{-2x} & -e^{3x} + 2e^{-2x} \end{bmatrix}.$$