Example 64. Let $p(D) = D^m + c_{m-1}D^{m-1} + \ldots + c_1 D + c_0$. Write the DE $p(D)y = 0$ as a system of (first-order) differential equations.

Solution. Write $y_k = D^k y$ for $k = 0, 1, \ldots, m-1$.

Then, $p(D)y = 0$ translates into the first-order system $\begin{cases} y_0' = y_1 \\ y_1' = y_2 \\ \vdots \\ y_{m-1}' = -c_{m-1}y_{m-1} - \ldots - c_1 y_1 - c_0 y_0 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & \cdots & \cdots & -c_{m-1} \end{bmatrix} \boldsymbol{y}$.

Comment. This is called the **companion matrix** of the polynomial $p(D)$. Can you see why the characteristic polynomial of the matrix must be (up to possibly a sign) equal to $p(D)$?

As expected, this works exactly the same way for recurrence equations:

Example 65. (extra) Let $p(N) = N^m + c_{m-1}N^{m-1} + \ldots + c_1 N + c_0$. Write the RE $p(N)a_n = 0$ as a system of (first-order) recurrences.

Solution. Write $a_n^{(k)} = N^k a_n = a_{n+k}$ for $k = 0, 1, \ldots, m-1$.

Then, $p(N)a_n = 0$ translates into the first-order system $\begin{cases} a_{n+1}^{(0)} = a_n^{(1)} \\ a_{n+1}^{(1)} = a_n^{(2)} \\ \vdots \\ a_{n+1}^{(m-1)} = -c_{m-1}a_n^{(m-1)} - \ldots - c_1 a_n^{(1)} - c_0 a_n^{(0)} \end{cases}$.

Let $\boldsymbol{a}_n = \begin{bmatrix} a_n^{(0)} \\ a_n^{(1)} \\ \vdots \\ a_n^{(m-1)} \end{bmatrix}$. Then, in matrix form, the RE is: $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ with $M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & \cdots & \cdots & -c_{m-1} \end{bmatrix}$

To solve $\boldsymbol{y}' = M\boldsymbol{y}$, determine the eigenvectors of M.

- Each λ-eigenvector $\boldsymbol{v}$ provides a solution: $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$
- If there is enough eigenvectors, these combine to the general solution.

Comment. If there is not enough eigenvectors, then we know what to do as well (at least in principle): instead of looking only for solutions of the type $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$, we also need to look for solutions of the type $\boldsymbol{y}(x) = (\boldsymbol{v}x + \boldsymbol{w})e^{\lambda x}$. Note that this can only happen if an eigenvalue is a repeated root of the characteristic polynomial.

Example 66. Let $M = \begin{bmatrix} -1 & 6 \\ -1 & 4 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

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Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M-\lambda I)=\det\left(\begin{bmatrix} -1-\lambda & 6 \\ -1 & 4-\lambda \end{bmatrix}\right)=(-1-\lambda)(4-\lambda)+6=\lambda^2-3\lambda+2=(\lambda-1)(\lambda-2)$

Hence, the eigenvalues are $\lambda=1$ and $\lambda=2$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda=1$, we need to solve $\begin{bmatrix} -2 & 6 \\ -1 & 3 \end{bmatrix}\boldsymbol{v}=\boldsymbol{0}$.
  Hence, $\boldsymbol{v}=\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda=1$.
- To find an eigenvector $\boldsymbol{v}$ for $\lambda=2$, we need to solve $\begin{bmatrix} -3 & 6 \\ -1 & 2 \end{bmatrix}\boldsymbol{v}=\boldsymbol{0}$.
  Hence, $\boldsymbol{v}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda=2$.

Hence, the general solution is $C_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^x+C_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{2x}$.

(b) The corresponding fundamental matrix solution is $\Phi=\begin{bmatrix} 3e^x & 2e^{2x} \\ e^x & e^{2x} \end{bmatrix}$.

(c) Note that $\Phi(0)=\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1}=\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$. It follows that

$$e^{Mx}=\Phi(x)\Phi(0)^{-1}=\begin{bmatrix} 3e^x & 2e^{2x} \\ e^x & e^{2x} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}=\begin{bmatrix} 3e^x-2e^{2x} & -6e^x+6e^{2x} \\ e^x-e^{2x} & -2e^x+3e^{2x} \end{bmatrix}.$$

(d) The solution to the IVP is $\boldsymbol{y}(x)=e^{Mx}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 3e^x-2e^{2x} & -6e^x+6e^{2x} \\ e^x-e^{2x} & -2e^x+3e^{2x} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} -3e^x+4e^{2x} \\ -e^x+2e^{2x} \end{bmatrix}$.

Note. If we hadn't already computed e^{Mx}, we would use the general solution and solve for the appropriate values of C_1 and C_2. Do it that way as well!

Theorem 67. Let M be $n\times n$. Then the **matrix exponential** satisfies

$$e^M=I+M+\frac{1}{2!}M^2+\frac{1}{3!}M^3+\ldots$$

Proof. Define $\Phi(x)=I+Mx+\frac{1}{2!}M^2x^2+\frac{1}{3!}M^3x^3+\ldots$

$$\begin{aligned}\Phi'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[I+Mx+\frac{1}{2!}M^2x^2+\frac{1}{3!}M^3x^3+\ldots\right] \\ &= 0+M+M^2x+\frac{1}{2!}M^3x^2+\ldots=M\Phi(x).\end{aligned}$$

Clearly, $\Phi(0)=I$. Therefore, $\Phi(x)$ is the fundamental matrix solution to $\boldsymbol{y}'=M\boldsymbol{y}$, $\boldsymbol{y}(0)=I$.
But that's precisely how we defined e^{Mx} earlier. It follows that $\Phi(x)=e^{Mx}$. □

(exponential function) e^x is the unique solution to $y'=y$, $y(0)=1$.
From here, it follows that $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$.

The latter is the Taylor series for e^x at $x=0$ that we have seen in Calculus II.
Important note. We can actually construct this infinite sum directly from $y'=y$ and $y(0)=1$.
Indeed, observe how each term, when differentiated, produces the term before it. For instance, $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3}{3!}=\frac{x^2}{2!}$.

Example 68. If $A=\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $A^{100}=\begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}$.

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Example 69. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2^2 & 0 \\ 0 & 5^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$.

Clearly, this works to obtain e^D for any diagonal matrix D.

In particular, for $Ax = \begin{bmatrix} 2x & 0 \\ 0 & 5x \end{bmatrix}$, $e^{Ax} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2x & 0 \\ 0 & 5x \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (2x)^2 & 0 \\ 0 & (5x)^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^{2x} & 0 \\ 0 & e^{5x} \end{bmatrix}$.

Example 70. Let $M = \begin{bmatrix} 8 & 4 \\ -1 & 4 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M - \lambda I) = \det\left(\begin{bmatrix} 8-\lambda & 4 \\ -1 & 4-\lambda \end{bmatrix}\right) = (8-\lambda)(4-\lambda) + 4 = \lambda^2 - 12\lambda + 36 = (\lambda - 6)(\lambda - 6)$

Hence, the eigenvalues are $\lambda = 6, 6$ (meaning that 6 has multiplicity 2).

- To find eigenvectors $\boldsymbol{v}$ for $\lambda = 6$, we need to solve $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 6$. There is no independent second eigenvector.

- We therefore search for a solution of the form $\boldsymbol{y}(x) = (\boldsymbol{v}x + \boldsymbol{w})e^{\lambda x}$ with $\lambda = 6$.

  $\boldsymbol{y}'(x) = (\lambda\boldsymbol{v}x + \lambda\boldsymbol{w} + \boldsymbol{v})e^{\lambda x} \overset{!}{=} M\boldsymbol{y} = (M\boldsymbol{v}x + M\boldsymbol{w})e^{\lambda x}$

  Equating coefficients of x, we need $\lambda\boldsymbol{v} = M\boldsymbol{v}$ and $\lambda\boldsymbol{w} + \boldsymbol{v} = M\boldsymbol{w}$.

  Hence, $\boldsymbol{v}$ must be an eigenvector (which we already computed); we choose $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

  [Note that any multiple of $\boldsymbol{y}(x)$ will be another solution, so it doesn't matter which multiple of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ we choose.]

  $\lambda\boldsymbol{w} + \boldsymbol{v} = M\boldsymbol{w}$ or $(M - \lambda)\boldsymbol{w} = \boldsymbol{v}$ then becomes $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\boldsymbol{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

  One solution is $\boldsymbol{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. [We only need one.]

Hence, the general solution is $C_1\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{6x} + C_2\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}x + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)e^{6x}$.

(b) The corresponding fundamental matrix solution is $\Phi = \begin{bmatrix} -2e^{6x} & -(2x+1)e^{6x} \\ e^{6x} & x\,e^{6x} \end{bmatrix}$.

(c) Note that $\Phi(0) = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$, so that $\Phi(0)^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} -2e^{6x} & -(2x+1)e^{6x} \\ e^{6x} & x\,e^{6x} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} (2x+1)e^{6x} & 4x\,e^{6x} \\ -x\,e^{6x} & -(2x-1)\,e^{6x} \end{bmatrix}.$$

(d) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (2x+1)e^{6x} & 4x\,e^{6x} \\ -x\,e^{6x} & -(2x-1)\,e^{6x} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (2x+1)e^{6x} \\ -x\,e^{6x} \end{bmatrix}$.