Let us revisit the inhomogeneous DE $my'' + ky = F(t)$ (which describes, for instance, the motion of a mass m on a spring with spring constant k under the influence of an external force $F(t)$).

We will solve the DE, for periodic forces $F(t)$, by using the Fourier series for $F(t)$. The same approach works likewise for linear equations of higher order, or even systems of equations.

Example 118. Find a particular solution of $2y'' + 32y = F(t)$, with $F(t) = \begin{cases} 10 & \text{if } t \in (0,1) \\ -10 & \text{if } t \in (1,2) \end{cases}$, extended 2-periodically.

Solution.

- From earlier, we already know $F(t) = 10\sum_{n \text{ odd}} \frac{4}{\pi n} \sin(\pi n t)$.
- We next solve the equation $2y'' + 32y = \sin(\pi n t)$ for $n = 1, 3, 5, \ldots$. First, we note that the external frequency is πn, which is never equal to the natural frequency $\omega_0 = 4$. Hence, there exists a particular solution of the form $y_p(t) = A\cos(\pi n t) + B\sin(\pi n t)$. To determine the coefficients A, B, we plug into the DE. Noting that $y_p'' = -\pi^2 n^2 y_p$ (why?!), we get

$$2y_p'' + 32y_p = (32 - 2\pi^2 n^2)(A\cos(\pi n t) + B\sin(\pi n t)) \stackrel{!}{=} \sin(\pi n t).$$

  We conclude $A = 0$ and $B = \frac{1}{32 - 2\pi^2 n^2}$, so that $y_p(t) = \frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}$.
- We combine the particular solutions found in the previous step, to see that

$$2y'' + 32y = 10\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\sin(\pi n t) \quad \text{is solved by} \quad y_p = 10\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n}\frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}.$$

Example 119. Find a particular solution of $2y'' + 32y = F(t)$, with $F(t)$ the 2π-periodic function such that $F(t) = 10t$ for $t \in (-\pi, \pi)$.

Solution.

- The Fourier series of $F(t)$ is $F(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{20}{n} \sin(nt)$. [Exercise!]
- We next solve the equation $2y'' + 32y = \sin(nt)$ for $n = 1, 2, 3, \ldots$. Note, however, that **resonance** occurs for $n = 4$, so we need to treat that case separately. If $n \neq 4$ then we find, as in the previous example, that $y_p(t) = \frac{\sin(nt)}{32 - 2n^2}$. [See how this fails for $n = 4$!]

  For $2y'' + 32y = \sin(4t)$, we begin with $y_p = At\cos(4t) + Bt\sin(4t)$. Then $y_p' = (A + 4Bt)\cos(4t) + (B - 4At)\sin(4t)$, and $y_p'' = (8B - 16At)\cos(4t) + (-8A - 16Bt)\sin(4t)$. Plugging into the DE, we get $2y_p'' + 32y_p = 16B\cos(4t) - 16A\sin(4t) \stackrel{!}{=} \sin(4t)$, and thus $B = 0$, $A = -\frac{1}{16}$. So, $y_p = -\frac{1}{16}t\cos(4t)$.
- We combine the particular solutions to get that our DE

$$2y'' + 32y = -5\sin(4t) + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty} (-1)^{n+1}\frac{20}{n}\sin(nt)$$

  is solved by

$$y_p(t) = \frac{5}{16}t\cos(4t) + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty} (-1)^{n+1}\frac{20}{n}\frac{\sin(nt)}{32 - 2n^2}.$$

  As in the previous example, this solution cannot really be simplified. Make some plots to appreciate the dominating character of the term resulting from resonance!

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Fourier cosine series and Fourier sine series

Suppose we have a function $f(t)$ which is defined on a finite interval $[0, L]$. Depending on the kind of application, we can extend $f(t)$ to a periodic function in three natural ways; in each case, we can then compute a Fourier series for $f(t)$ (which will agree with $f(t)$ on $[0, L]$).

Comment. Here, we do not worry about the definition of $f(t)$ at specific individual points like $t=0$ and $t=L$, or at jump discontinuities. Recall that, at a discontinuity, a Fourier series takes the average value.

(a) We can extend $f(t)$ to an L-periodic function.

In that case, we obtain the Fourier series $f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{2\pi n t}{L}\right)+b_n\sin\left(\frac{2\pi n t}{L}\right)\right)$.

(b) We can extend $f(t)$ to an even $2L$-periodic function.

In that case, we obtain the **Fourier cosine series** $f(t)=\frac{\tilde{a}_0}{2}+\sum_{n=1}^{\infty}\tilde{a}_n\cos\left(\frac{\pi n t}{L}\right)$.

(c) We can extend $f(t)$ to an odd $2L$-periodic function.

In that case, we obtain the **Fourier sine series** $f(t)=\sum_{n=1}^{\infty}\tilde{b}_n\sin\left(\frac{\pi n t}{L}\right)$.

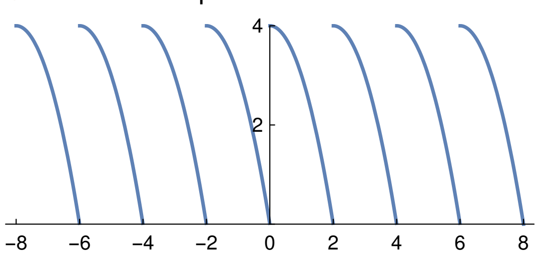
Example 120. Consider the function $f(t)=4-t^2$, defined for $t\in[0,2]$.

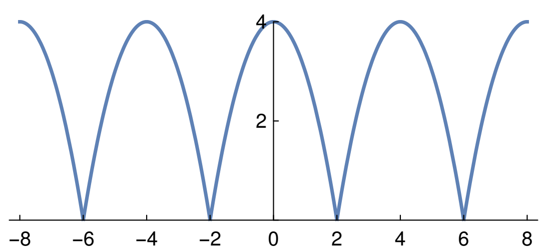
(a) Sketch the 2-periodic extension of $f(t)$.

(b) Sketch the 4-periodic even extension of $f(t)$.

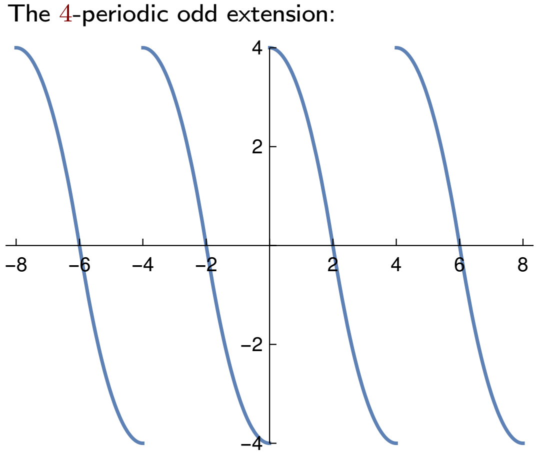
(c) Sketch the 4-periodic odd extension of $f(t)$.

Solution. The 2-periodic extension as well as the 4-periodic even extension:





The 4-periodic odd extension:



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Boundary value problems and partial differential equations

Endpoint problems and eigenvalues

Example 121. The **IVP** (initial value problem) $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = 0$ has the unique solution $y(x) = 0$.

Initial value problems are often used when the problem depends on time. Then, $y(0)$ and $y'(0)$ describe the initial configuration at $t = 0$.

For problems which instead depend on spatial variables, such as position, it may be natural to specify values at positions on the boundary (for instance, if $y(x)$ describes the steady-state temperature of a rod at position x, we might know the temperature at the two end points).

The next two examples illustrate that such boundary value problem may or may not have unique solutions.

Example 122. The **BVP** (boundary value problem) $y'' + 4y = 0$, $y(0) = 0$, $y(1) = 0$ has the unique solution $y(x) = 0$.

We know that the general solution to the DE is $y(x) = A\cos(2x) + B\sin(2x)$. The boundary conditions imply $y(0) = A \overset{!}{=} 0$ and, already using that $A = 0$, $y(1) = B\sin(2) \overset{!}{=} 0$ shows that $B = 0$ as well.

Example 123. The BVP $y'' + \pi^2 y = 0$, $y(0) = 0$, $y(1) = 0$ is solved by $y(x) = B\sin(\pi x)$ for any value B.

This time, the general solution to the DE is $y(x) = A\cos(\pi x) + B\sin(\pi x)$. The boundary conditions imply $y(0) = A \overset{!}{=} 0$ and, using that $A = 0$, $y(1) = B\sin(\pi) \overset{!}{=} 0$. This second condition true for any B.