Review.

- The BVP $y''+4y=0,\ y(0)=0,\ y(1)=0$ has the unique solution $y(x)=0$.
- The BVP $y''+\pi^2y=0,\ y(0)=0,\ y(1)=0$ is solved by $y(x)=B\sin(\pi x)$ for any value B.

It is therefore natural to ask: for which λ does the BVP $y''+\lambda y=0,\ y(0)=0,\ y(L)=0$ have nonzero solutions?

Such solutions are called **eigenfunctions** and λ is the corresponding **eigenvalue**.

Remark. Compare that to our previous use of the term eigenvalue: given a matrix A, we asked: for which λ does $A\boldsymbol{v}-\lambda\boldsymbol{v}=0$ have nonzero solutions $\boldsymbol{v}$? Such solutions were called eigenvectors and λ was the corresponding eigenvalue.

Example 124. Find all eigenfunctions and eigenvalues of $y''+\lambda y=0,\ y(0)=0,\ y(L)=0$.

Such a problem is called an **eigenvalue problem**.

Solution. The solutions of the DE look different in the cases $\lambda<0$, $\lambda=0$, $\lambda>0$, so we consider them individually.

$\boldsymbol{\lambda=0}$. Then $y(x)=Ax+B$ and $y(0)=y(L)=0$ implies that $y(x)=0$. No eigenfunction here.

$\boldsymbol{\lambda<0}$. Write $\lambda=-\rho^2$. Then $y(x)=Ae^{\rho x}+Be^{-\rho x}$. $y(0)=A+B\overset{!}{=}0$ implies $B=-A$. Using that, we get $y(L)=A(e^{\rho L}-e^{-\rho L})\overset{!}{=}0$. For eigenfunctions we need $A\neq 0$, so $e^{\rho L}=e^{-\rho L}$ which implies $\rho L=-\rho L$. This cannot happen since $\rho\neq 0$ and $L\neq 0$. Again, no eigenfunctions in this case.

$\boldsymbol{\lambda>0}$. Write $\lambda=\rho^2$. Then $y(x)=A\cos(\rho x)+B\sin(\rho x)$. $y(0)=A\overset{!}{=}0$. Using that, $y(L)=B\sin(\rho L)\overset{!}{=}0$. Since $B\neq 0$ for eigenfunctions, we need $\sin(\rho L)=0$. This happens if $\rho L=n\pi$ for $n=0,1,2,\ldots$. Consequently, we do find the eigenfunctions $y_n(x)=\sin\frac{n\pi x}{L}$, $n=1,2,3,\ldots$, with eigenvalue $\lambda=\left(\frac{n\pi}{L}\right)^2$.

Example 125. Suppose that a rod of length L is compressed by a force P (with ends being pinned [not clamped] down). We model the shape of the rod by a function $y(x)$ on some interval $[0,L]$.

The theory of elasticity predicts that, under certain simplifying assumptions, y should satisfy $EIy''+Py=0,\ y(0)=0,\ y(L)=0$.

Here, EI is a constant modeling the inflexibility of the rod (E, known as Young's modulus, depends on the material, and I depends on the shape of cross-sections (it is the area moment of inertia)).

In other words, $y''+\lambda y=0,\ y(0)=0,\ y(L)=0$, with $\lambda=\frac{P}{EI}$.

The fact that there is no nonzero solution unless $\lambda=\left(\frac{\pi n}{L}\right)^2$ for some $n=1,2,3,\ldots$, means that buckling can only occur if $P=\left(\frac{\pi n}{L}\right)^2EI$. In particular, no buckling occurs for forces less than $\frac{\pi^2EI}{L^2}$. This is known as the critical load (or Euler load) of the rod.

Comment. This is a very simplified model. In particular, it assumes that the deflections are small. (Technically, the buckled rod in our model is longer than L; of course, that's not the case in practice.)

https://en.wikipedia.org/wiki/Euler%27s_critical_load

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The heat equation

We wish to describe one-dimensional heat flow.

Comment. If this sounds very specialized, it might help to know that the heat equation is also used, for instance, in probability (Brownian motion), financial math (Black-Scholes), or chemical processes (diffusion equation).

Let $u(x,t)$ describe the temperature at time t at position x.

If we model a heated rod of length L, then $x \in [0, L]$.

Notation. $u(x,t)$ depends on two variables. When taking derivatives, we will use the notations $u_t = \frac{\partial}{\partial t} u$ and $u_{xx} = \frac{\partial^2}{\partial x^2} u$ for first and higher derivatives.

Experience tells us that heat flows from warmer to cooler areas and has an averaging effect.

Make a sketch of some temperature profile $u(x,t)$ for fixed t.

As t increases, we expect maxima (where $u_{xx} < 0$) of that profile to flatten out (which means that $u_t < 0$); similarly, minima (where $u_{xx} > 0$) should go up (meaning that $u_t > 0$). The simplest relationship between u_t and u_{xx} which conforms with our expectation is $u_t = k u_{xx}$, with $k > 0$.

(heat equation)

$$u_t = k u_{xx}$$

Note that the heat equation is a linear and homogeneous **partial differential equation**.

In particular, the principle of superposition holds: if u_1 and u_2 solve the heat equation, then so does $c_1 u_1 + c_2 u_2$.

Higher dimensions. In higher dimensions, the heat equation takes the form $u_t = k(u_{xx} + u_{yy})$ or $u_t = k(u_{xx} + u_{yy} + u_{zz})$. Note that $\Delta u = u_{xx} + u_{yy} + u_{zz}$ is the Laplace operator you may know from Calculus III.

The Laplacian Δu is also often written as $\Delta u = \nabla^2 u$. The operator $\nabla = (\partial / \partial x, \partial / \partial y)$ is pronounced "nabla" (Greek for a certain harp) or "del" (Persian for heart), and ∇^2 is short for the inner product $\nabla \cdot \nabla$.

Example 126. Note that $u(x,t) = ax + b$ solves the heat equation.

Example 127. To get a feeling, let us find some other solutions to $u_t = u_{xx}$ (for starters, $k = 1$).

- For instance, $u(x,t) = e^t e^x$ is a solution.
  [Not a very interesting one for modeling heat flow because it increases exponentially in time.]
- ... to be continued ...
  Can you find further solutions?

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