The boundary conditions in the next example model insulated ends.

Example 131. Find the unique solution $u(x,t)$ to:

$$\begin{aligned} &u_t = k u_{xx} && \text{(PDE)} \\ &u_x(0,t) = u_x(L,t) = 0 && \text{(BC)} \\ &u(x,0) = f(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

Solution.

- We proceed as before and look for solutions $u(x,t) = X(x)T(t)$ (**separation of variables**).
  Plugging into (PDE), we get $X(x)T'(t) = kX''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{const} =: -\lambda$.
  We thus have $X'' + \lambda X = 0$ and $T' + \lambda k T = 0$.
- From the (BC), i.e. $u_x(0,t) = X'(0)T(t) = 0$, we get $X'(0) = 0$.
  Likewise, $u_x(L,t) = X'(L)T(t) = 0$ implies $X'(L) = 0$.
- So X solves $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(L) = 0$. It is left as a homework to show that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x) = \cos(\frac{\pi n}{L}x)$ corresponding to $\lambda = (\frac{\pi n}{L})^2$, $n = 0, 1, 2, 3....$ [See practice problems.]
- On the other hand (as before), T solves $T' + \lambda k T = 0$, and hence $T(t) = e^{-\lambda k t} = e^{-(\frac{\pi n}{L})^2 k t}$.
- Taken together, we have the solutions $u_n(x,t) = e^{-(\frac{\pi n}{L})^2 k t} \cos(\frac{\pi n}{L}x)$ solving $\text{(PDE)}+\text{(BC)}$.
- We wish to combine these in such a way that (IC) holds.
  At $t = 0$, $u_n(x,0) = \cos(\frac{\pi n}{L}x)$. All of these are $2L$-periodic.
  Hence, we extend $f(x)$, which is only given on $(0,L)$, to an even $2L$-periodic function (its Fourier cosine series!). By making it even, its Fourier series only involves cosine terms: $f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos(\frac{\pi n}{L}x)$.
  Note that

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x,$$

  where the first integral makes reference to the extension of $f(x)$ while the second integral only uses $f(x)$ on its original interval of definition.

Consequently, $\text{(PDE)}+\text{(BC)}+\text{(IC)}$ is solved by

$$u(x,t) = \frac{a_0}{2}u_0(x,t) + \sum_{n=1}^{\infty} a_n u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-(\frac{\pi n}{L})^2 k t}\cos\left(\frac{\pi n}{L}x\right),$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

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The inhomogeneous heat equation

We next indicate that we can similarly solve the nonhomogeneous heat equation (with nonhomogeneous boundary conditions).

Comment. We indicated earlier that

$$\begin{aligned} &u_t = k u_{xx} && \text{(PDE)} \\ &u(0,t) = a, \quad u(L,t) = b && \text{(BC)} \\ &u(x,0) = f(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

can be solved by realizing that $Ax + B$ solves (PDE).

Indeed, let $v(x) = a + \frac{b-a}{L}x$ (so that $v(0) = a$ and $v(L) = b$). We then look for a solution of the form $u(x,t) = v(x) + w(x,t)$. Note that $u(x,t)$ solves (PDE)+(BC)+(IC) if and only if $w(x,t)$ solves:

$$\begin{aligned} &w_t = k w_{xx} && \text{(PDE)} \\ &w(0,t) = 0, \quad w(L,t) = 0 && \text{(BC*)} \\ &w(x,0) = f(x) - v(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

This the (homogeneous) heat equation that we know how to solve.

$v(x)$ is called the **steady-state solution** (it does not depend on time!) and $w(x,t)$ the **transient solution** (note that $w(x,t)$ and its partial derivatives tend to zero as $t \to \infty$).

Example 132. Consider the heat flow problem:

$$\begin{aligned} &u_t = 3u_{xx} + 4x^2 && \text{(PDE)} \\ &u(0,t) = 1, \quad u_x(3,t) = -5 && \text{(BC)} \\ &u(x,0) = f(x), \quad x \in (0,3) && \text{(IC)} \end{aligned}$$

Determine the steady-state solution and spell out equations characterizing the transient solution.

Solution. We look for a solution of the form $u(x,t) = v(x) + w(x,t)$, where $v(x)$ is the steady-state solution and the transient solution $w(x,t)$ (as well as its derivatives) tend to zero as $t \to \infty$.

- Plugging into (PDE), we get $w_t = 3v'' + 3w_{xx} + 4x^2$. Letting $t \to \infty$, this becomes $0 = 3v'' + 4x^2$.
- Plugging into (BC), we get $w(0,t) + v(0) = 1$ and $w_x(3,t) + v'(3) = -5$.
  Letting $t \to \infty$, this becomes $v(0) = 1$ and $v'(3) = -5$.
- Solving the ODE $0 = 3v'' + 4x^2$ with boundary conditions $v(0) = 1$ and $v'(3) = -5$, we find

  $$v(x) = \iint -\frac{4}{3}x^2 \mathrm{d}x \mathrm{d}x = -\frac{1}{9}x^4 + C_1 + C_2 x$$

  and therefore the steady-state solution $v(x) = -\frac{1}{9}x^4 + 1 + 7x$.

On the other hand, the transient solution $w(x,t)$ is characterized as the unique solution to:

$$\begin{aligned} &w_t = 3w_{xx} && \text{(PDE*)} \\ &w(0,t) = 0, \quad w_x(3,t) = 0 && \text{(BC*)} \\ &w(x,0) = f(x) - v(x) && \text{(IC*)} \end{aligned}$$

We know how to solve this homogeneous heat flow problem (see practice problems) using separation of variables.

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