Sketch of Lecture 25

Tue, 12/3/2019

Finite difference method

We know from Calculus that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

PDEs quickly become impossibly difficult to approach with exact solution techniques.

It is common therefore to proceed numerically. One approach is to discretize the problem.

For instance. We could use $f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$ to replace $f'(x)$ with the **finite difference** on the RHS.

Such approximate methods are called **finite difference methods**.

These days, finite difference methods are the most common approach to numerically solving PDEs.
The ODE or PDE translates into a (sparse) system of linear equations which is solved using Linear Algebra.

Example 137.

- $f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$ is a **forward difference** for $f'(x)$.
- $f'(x) \approx \frac{1}{h}[f(x) - f(x-h)]$ is a **backward difference** for $f'(x)$.
- $f'(x) \approx \frac{1}{h}\left[f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)\right]$ is a **central difference** for $f'(x)$.

Comment. Recall that power series $f(x)$ have the Taylor expansion $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$.

Equivalently, $f(x+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} h^n = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + O(h^4)$. It follows that

$$\frac{1}{h}[f(x+h) - f(x)] = f'(x) + \boxed{\frac{h}{2} f''(x) + O(h^2)} = f'(x) + \boxed{O(h)}.$$

The $\boxed{\text{error}}$ is of order $O(h)$. On the other hand,

$$\begin{aligned} f\left(x + \frac{h}{2}\right) &= f(x) + \frac{h}{2} f'(x) + \frac{h^2}{8} f''(x) + \frac{h^3}{48} f'''(x) + O(h^4), \\ f\left(x - \frac{h}{2}\right) &= f(x) - \frac{h}{2} f'(x) + \frac{h^2}{8} f''(x) - \frac{h^3}{48} f'''(x) + O(h^4), \end{aligned}$$

from which it follows that

$$\frac{1}{h}\left[f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)\right] = f'(x) + \boxed{\frac{h^2}{24} f'''(x) + O(h^3)} = f'(x) + \boxed{O(h^2)}.$$

The $\boxed{\text{error}}$ is of order $O(h^2)$.

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Example 138. Find a central difference for $f''(x)$ and determine the order of the error.

Solution. $f''(x) \approx \frac{1}{h}\left[f'\left(x+\frac{h}{2}\right) - f'\left(x-\frac{h}{2}\right)\right] \approx \frac{1}{h^2}[f(x+h) - 2f(x) + f(x-h)]$

Using

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) + O(h^5), \\ f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) + O(h^5), \end{aligned}$$

we find that

$$\frac{1}{h^2}[f(x+h) - 2f(x) + f(x-h)] = f''(x) + \boxed{\frac{h^2}{12}f^{(4)}(x) + O(h^3)} = f''(x) + \boxed{O(h^2)}.$$

The $\boxed{\text{error}}$ is of order $O(h^2)$.

Example 139. (discretizing Δ) Use the previous example to derive a finite difference for $\Delta u = u_{xx} + u_{yy}$ in 2D.

Solution. $\Delta u \approx \frac{1}{h^2}[u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y)]$

Notation. This finite difference is typically represented as $\frac{1}{h^2}\left[\begin{array}{ccc} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{array}\right]$, the **five-point stencil**.

Comment. Recall that solutions to $\Delta u = 0$ describe steady-state temperature distributions. Can you see how our finite difference can be connected to the exchange of heat?

Comment. Think about how to use this finite difference to numerically solve the corresponding Dirichlet problem by discretizing (one equation per lattice point).

Advanced comment. If $\Delta u = 0$ then, when discretizing, the center point has the average value of the four points adjacent to it. This leads to the **maximum principle**: if $\Delta u = 0$ on a region R, then the maximum (and, likewise, minimum) value of u must occur at a boundary point of R.

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