Linear first-order DEs

DE $\quad y' = a(x)\,y + f(x)$

corresponding homogeneous DE:

HDE $\quad y' = a(x)\,y$

EG (previously) $\quad y' = x^2 y$

general solution: $y_h = C e^{\frac{1}{3}x^3}$

HDE $\quad y' = a(x)\,y$

general solution: $y_h = C e^{\int a(x)\,dx}$

DE $\quad y' = a(x)\,y + f(x)$

particular solution: $y_p = y_h(x) \int \frac{f(x)}{y_h(x)}\,dx$

Why? variation of constants

look for $y_p(x) = c(x)\, y_h(x)$

plug into DE: $\quad y_p' = a\, y_p + f$

$$c' y_h + c\underbrace{y_h'}_{\text{HDE } = a y_h} = a c y_h + f$$

$$c' y_h = f \qquad c' = \frac{f}{y_h}$$

$$c(x) = \int \frac{f(x)}{y_h(x)}\,dx$$

EG $\quad x^2 y' = 1 - xy + 2x \qquad y(1) = 3$

$$y' = \underbrace{-\frac{1}{x}}_{a(x)} y + \underbrace{\frac{1}{x^2} + \frac{2}{x}}_{f(x)}$$

x positive

$$y_h(x) = e^{\int a(x)\,dx} = e^{\int -\frac{1}{x}\,dx} = e^{-\ln|x|} = e^{-\ln x} = \frac{1}{x}$$

$$y_p(x) = y_h(x) \int \frac{f(x)}{y_h(x)}\,dx = \frac{1}{x} \int \frac{\frac{1}{x^2} + \frac{2}{x}}{\frac{1}{x}}\,dx = \frac{1}{x} \int \left(\frac{1}{x} + 2\right) dx$$

$$= \frac{1}{x}\left(\ln x + 2x + C\right) \qquad \text{general solution}$$

set $x=1$: $\quad 2 + C = 3 \rightsquigarrow C = 1$

$$\Rightarrow \quad y = \frac{1}{x}\left(\ln x + 2x + 1\right)$$