Differential operators

LDE: $Ly = f(x)$

where $\underbrace{L = D^n + p_{n-1}(x) D^{n-1} + \dots + p_1(x) D + p_0(x)}_{\text{differential operator in normal form}}$

if constant $\Rightarrow$ know how to solve

EG xD is in normal form
$\neq$ Dx is not in normal form

$(xD)\, y(x) = x y'(x)$

$(Dx)\, y(x) = \frac{d}{dx} x y(x) = y(x) + x y'(x)$

$= (1 + xD)\, y(x)$

$\Longrightarrow Dx = 1 + xD$ is in normal form

EG $D^2 a$ with $a = a(x)$
write in normal form

$(D^2 a)\, y = \frac{d^2}{dx^2} a y = \frac{d}{dx}\left[a'y + ay'\right]$

$= \left[a''y + \underbrace{a'y' + a'y'}_{2a'y'} + ay''\right]$

$= (aD^2 + 2a'D + a'')\, y$

$\Longrightarrow D^2 a = aD^2 + 2a'D + a''$ is in normal form

EG Normal form of $(D+a)(D+b)$ $\quad a = a(x)$, $b = b(x)$

$(D+a)(D+b) = D^2 + \underbrace{Db}_{= bD + b'} + aD + ab$

$= D^2 + (a+b)D + (ab + b')$
is in normal form

$(Db)y$
$= b'y + by'$
$= (bD + b')y$

EG $(D + \underbrace{\tfrac{1}{x}}_{a})(D - \underbrace{\tfrac{1}{x}}_{b}) = D^2 = D \cdot D$

i.e. factorization is not unique