Inhomogeneous systems of DEs

$$y' = A(x)y + f(x)$$

has particular solution

$$y_p(x) = \Phi(x) \int \Phi^{-1}(x) f(x)\,dx$$

where

$\Phi(x)$ is any fundamental matrix solution to

$$y' = A(x)y$$

review $y' = a(x)y + f(x)$

has particular solution:

$$y_p(x) = y_h(x) \int \frac{f(x)}{y_h(x)}\,dx$$

where

y_h solves $y' = a(x)y$

variation of constants

nice case if $\Phi(x) = e^{Ax}$ then $\Phi^{-1}(x) = e^{-Ax}$

$A(x) = A$

- $y' = Ay \quad y(0) = c \implies y(x) = e^{Ax}c$
- $y' = Ay + f(x) \quad y(0) = c \implies y(x) = \underbrace{e^{Ax}c}_{y_h} + \underbrace{e^{Ax}\int_0^x e^{-At} f(t)\,dt}_{y_p}$

EG Suppose $e^{Ax} = \begin{bmatrix} 2e^{2x} - e^{3x} & -2e^{2x} + 2e^{3x} \\ e^{2x} - e^{3x} & -e^{2x} + 2e^{3x} \end{bmatrix}$.

check: $x = 0 \rightsquigarrow \begin{bmatrix} 2-1 & -2+2 \\ 1-1 & -1+2 \end{bmatrix} = I$

(a) Solve $y' = Ay$, $y(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$\implies y(x) = e^{Ax}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2e^{2x} + 3e^{3x} \\ -e^{2x} + 3e^{3x} \end{bmatrix}$$

(b) Solve $y' = Ay + \begin{bmatrix} 0 \\ 2e^x \end{bmatrix}$, $y(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

check: $x = 0 \checkmark \rightsquigarrow \begin{bmatrix} 2-6+5 \\ -3+5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\implies y(x) = e^{Ax}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{Ax}\int_0^x e^{-At} f(t)\,dt = \begin{bmatrix} 2e^x - 6e^{2x} + 5e^{3x} \\ -3e^{2x} + 5e^{3x} \end{bmatrix}$$

$$\int_0^x e^{at}\,dt = \left[\frac{1}{a}e^{at}\right]_0^x = \frac{1}{a}e^{ax} - \frac{1}{a}$$

$x = 0 \rightsquigarrow \frac{1}{a} - \frac{1}{a} = 0$

$$\int_0^x \begin{bmatrix} 2e^{-2t} - e^{-3t} & -2e^{-2t} + 2e^{-3t} \\ e^{-2t} - e^{-3t} & -e^{-2t} + 2e^{-3t} \end{bmatrix}\begin{bmatrix} 0 \\ 2e^t \end{bmatrix} dt$$

$$= \int_0^x \begin{bmatrix} -4e^{-t} + 4e^{-2t} \\ -2e^{-t} + 4e^{-2t} \end{bmatrix} dt$$

$$= \begin{bmatrix} 4e^{-x} - 2e^{-2x} - 2 \\ 2e^{-x} - 2e^{-2x} \end{bmatrix}$$

(c) What is A?

$$A = \left[\underbrace{\frac{d}{dx}e^{Ax}}_{Ae^{Ax}}\right]_{x=0} = \begin{bmatrix} 4e^{2x} - 3e^{3x} & -4e^{2x} + 6e^{3x} \\ 2e^{2x} - 3e^{3x} & -2e^{2x} + 6e^{3x} \end{bmatrix}_{x=0} = \begin{bmatrix} \overset{4-3}{1} & 2 \\ -1 & 4 \end{bmatrix}$$