Power series + non-linear DEs

EG $y'' = \cos(x+y) \qquad y(0)=0, \quad y'(0)=1$

Determine first few terms of the power series for $y(x)$.
(around $x=0$)

options
(A) successive differentiation (by hand)
(B) plug in power series (by computer)

(A)

$$y'' = \cos(x+y) \qquad y''(0) = \cos(0+\underbrace{y(0)}_{=0}) = \cos(0) = 1$$

$$y''' = -\sin(x+y)(1+y') \qquad y'''(0) = -\underbrace{\sin(0)}_{=0}(1+y'(0)) = 0$$

$$y^{(4)} = -\cos(x+y)(1+y')^2 - \sin(x+y)\,y'' \qquad y^{(4)}(0) = -\cos(0)(1+\underbrace{y'(0)}_{=1})^2 - \underbrace{\sin(0)}_{=0}\,y''(0) = -4$$

$$y^{(5)} = \ldots \qquad y^{(5)}(0) = \ldots = -6$$

$$\Longrightarrow \quad y = \underbrace{y(0)}_{0} + \underbrace{y'(0)}_{1}x + \frac{\overbrace{y''(0)}^{1}}{2}x^2 + \frac{\overbrace{y'''(0)}^{0}}{6}x^3 + \frac{\overbrace{y^{(4)}(0)}^{-4}}{24}x^4 + \frac{\overbrace{y^{(5)}(0)}^{-6}}{120}x^5 + \ldots$$

$$= x + \frac{1}{2}x^2 - \frac{1}{6}x^4 - \frac{1}{20}x^5 + \ldots$$

(B)

$$y = x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots$$

$$y' = 1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \ldots$$

compare

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots$$

$$\cos(x+y) = 1 - \frac{1}{2}(x+y)^2 + \frac{1}{24}(x+y)^4 - \ldots$$

not needed here

$$= 1 - \frac{1}{2}x^2 - xy - \frac{1}{2}y^2 + \ldots$$

$$(x + a_2x^2 + a_3x^3 + \ldots)^2 = x^2 + 2a_2x^3 + O(x^4)$$

$$= 1 - \frac{1}{2}x^2 - x(x + a_2x^2 + \ldots) - \frac{1}{2}(x^2 + 2a_2x^3 + \ldots) + \ldots$$

$$= 1 - 2x^2 - 2a_2x^3 + \ldots$$

compare with y'':

$$2a_2 = 1 \qquad a_2 = \frac{1}{2}$$
$$6a_3 = 0 \qquad a_3 = 0$$
$$12a_4 = -2 \qquad a_4 = -\frac{1}{6}$$
$$20a_5 = -2a_2 \qquad a_5 = -\frac{1}{10}a_2 = -\frac{1}{20}$$