Resonance

idea: Fourier series

recall $my'' + ky = F(t)$

models motion of a mass m on a spring with spring constant k under the influence of an external force $F(t)$

EG $my'' + ky = \cos(\omega t)$

$\omega > 0$

When does resonance occur?

$mD^2 + k$

"old" roots : $\pm i\sqrt{\frac{k}{m}}$ — natural frequency ω_0

"new" roots : $\pm i\omega$ — external frequency

$my'' + ky = 0$

$A\cos(\omega_0 t) + B\sin(\omega_0 t)$

resonance $\iff$ $\omega = \omega_0$

$\omega = \sqrt{\frac{k}{m}}$

EG $2y'' + 32y = \sum_{n=1}^{\infty} \frac{\cos(n\omega t)}{n^2+1}$

When does resonance occur?

"old" roots : $\pm i\sqrt{\frac{32}{2}} = \pm 4i$ natural frequency $= 4$

resonance $\iff$ $n\omega = 4$

$\omega = \frac{4}{n}$ for some $n = 1, 2, 3, \ldots$

EG $2y'' + 32y = F(t)$

$F(t)$ 2π-periodic with $F(t) = 10t$ for $t \in (-\pi, \pi)$

odd!

① Fourier series for $F(t)$: $F(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{20}{n} \sin(nt)$

② solve : $2y'' + 32y = \sin(nt)$

"old" roots : $\pm 4i$ "new roots" : $\pm in$

resonance for $n = 4$

$n \neq 4$ $y_p = A\cos(nt) + B\sin(nt) \underset{\text{do it!}}{=} \frac{\sin(nt)}{32 - 2n^2}$

$n = 4$ $y_p = At\cos(4t) + Bt\sin(4t) \underset{\text{do it!}}{=} -\frac{1}{16} t\cos(4t)$

③ $2y'' + 32y$

$= -5\sin(4t) + \sum_{\substack{n=1 \\ n\neq 4}}^{\infty} (-1)^{n+1} \frac{20}{n} \sin(nt)$

solved by : $\boxed{\frac{5}{16} t\cos(4t) + \sum_{\substack{n=1 \\ n\neq 4}}^{\infty} (-1)^{n+1} \frac{20}{n} \frac{\sin(nt)}{32 - 2n^2}}$