Boundary value problems

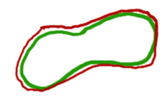
initial value problem $\quad y'' + 4y = 0 \quad y(0) = 0, \; y'(0) = 0$

unique solution : $y(x) = 0$

boundary value problem $\quad y'' + 4y = 0 \quad y(0) = 0, \; y(1) = 0$

boundary conditions

1D


2D


general sol. to DE:

$y(x) = A\cos(\underset{\pi}{2}x) + B\sin(\underset{\pi}{2}x)$

$y(0) = 0 = A \cdot 1 \rightsquigarrow A = 0$

$y(1) = B\underbrace{\sin(\overset{\pi}{2})}_{\neq 0} = 0 \rightarrow B = 0$

$\sin(\pi) = 0$

unique solution: $y(x) = 0$

$y'' + \pi^2 y = 0 \quad y(0) = 0, \; y(1) = 0$

solutions: $y(x) = B\sin(\pi x)$

EG $\quad y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$

eigenvalue problem

Find all eigenfunctions + eigenvalues.

previously:

$Av = \lambda v$

$Av - \lambda v = 0$

char. poly: $D^2 + \lambda \quad$ roots : $\pm\sqrt{-\lambda}$

$\lambda = 0$ $\quad y(x) = A + Bx$

$y(0) = A = 0 \qquad y(L) = B\underset{\neq 0}{L} = 0 \rightarrow B = 0$

$\Rightarrow$ only trivial solution $y(x) = 0$

$\lambda < 0$ $\quad p = \sqrt{-\lambda} \qquad y(x) = Ae^{px} + Be^{-px}$

$y(0) = A + B = 0 \rightsquigarrow B = -A$

$y(L) = \underbrace{Ae^{pL} - Ae^{-pL}}_{= A\underbrace{(e^{pL} - e^{-pL})}_{\neq 0}} = 0$

need: $e^{pL} - e^{-pL} = 0$

$e^{pL} = e^{-pL}$

$p\underset{\neq 0}{\cancel{L}} = -p\cancel{L}$

$p = 0 \quad ↯$

$\Leftrightarrow \lambda = 0$

$\rightsquigarrow A = 0$

$\rightarrow B = 0$

$\Rightarrow$ only trivial solution $y(x) = 0$

$\lambda > 0$ roots: $\pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}$ $\quad p = \sqrt{\lambda}$ $\quad \lambda = p^2$

$$y(x) = A\cos(px) + B\sin(px)$$

$$y(0) = A\underbrace{\cos(0)}_{=1} = 0 \rightsquigarrow A = 0$$

$$y(L) = B\underbrace{\sin(pL)}_{} = 0$$

need: $\sin(pL) = 0$

$\iff \underbrace{pL}_{>0} = n\pi$ for some $n = 1, 2, 3, \dots$

$\iff p = \frac{n\pi}{L}$

$\iff \lambda = \left(\frac{n\pi}{L}\right)^2$

solutions:

$$y(x) = \cancel{B}\sin\left(\frac{n\pi}{L}x\right) \quad \text{for} \quad \lambda = \left(\frac{n\pi}{L}\right)^2$$

eigenfunctions $\qquad$ eigenvalues

alternative boundary conditions:

$$y(0) = a \qquad y(L) = b$$

or: $y(0) = a \qquad y'(L) = b$