Example 29. (review) Find the general solution of $y''' - y'' - 5y' - 3y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - D^2 - 5D - 3 = (D-3)(D+1)^2$ has roots $3, -1, -1$. Hence, the general solution is $y(x) = C_1 e^{3x} + (C_2 + C_3 x) e^{-x}$.

Example 30. Find the general solution of $y'' + 4y = 12x$.

Solution. Here, $p(D) = D^2 + 4$, which has roots $\pm 2i$.

Hence, the general solution is $y(x) = y_p(x) + C_1 \cos(2x) + C_2 \sin(2x)$. It remains to find a particular solution y_p.

Noting that $D^2 \cdot (12x) = 0$, we apply D^2 to both sides of the DE.

We get $D^2(D^2+4) \cdot y = 0$, which is a homogeneous linear DE! Its general solution is $C_1 + C_2 x + C_3 \cos(2x) + C_4 \sin(2x)$. In particular, y_p is of this form for some choice of $C_1, \ldots, C_4$.

It simplifies our life to note that there has to be a particular solution of the simpler form $y_p = C_1 + C_2 x$.

[Why?! Because we know that $C_3 \cos(2x) + C_4 \sin(2x)$ can be added to any particular solution.]

It only remains to find appropriate values C_1, C_2 such that $y_p'' + 4y_p = 12x$. Since $y_p'' + 4y_p = 4C_1 + 4C_2 x$, comparing coefficients yields $4C_1 = 0$ and $4C_2 = 12$, so that $C_1 = 0$ and $C_2 = 3$. In other words, $y_p = 3x$.

Therefore, the general solution to the original DE is $y(x) = 3x + C_1 \cos(2x) + C_2 \sin(2x)$.

Example 31. Find the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. This is $p(D) y = e^{3x}$ with $p(D) = D^2 + 4D + 4 = (D+2)^2$.

Hence, the general solution is $y(x) = y_p(x) + (C_1 + C_2 x) e^{-2x}$. It remains to find a particular solution y_p.

Note that $(D-3) e^{3x} = 0$. Hence, we apply $(D-3)$ to the DE to get $(D-3)(D+2)^2 y = 0$.

This homogeneous linear DE has general solution $(C_1 + C_2 x) e^{-2x} + C_3 e^{3x}$. We conclude that the original DE must have a particular solution of the form $y_p = C_3 e^{3x}$.

To determine the value of C_3, we plug into the original DE: $y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4) C_3 e^{3x} \stackrel{!}{=} e^{3x}$. Hence, $C_3 = 1/25$. In conclusion, the general solution is $y(x) = (C_1 + C_2 x) e^{-2x} + \frac{1}{25} e^{3x}$.

We found a recipe for solving nonhomogeneous linear DEs with constant coefficients.

Our approach works for $p(D) y = f(x)$ whenever the right-hand side $f(x)$ is the solution of some homogeneous linear DE with constant coefficients: $q(D) f(x) = 0$

Theorem 32. To find a particular solution y_p to an inhomogeneous linear DE with constant coefficients $p(D) y = f(x)$:

- Find $q(D)$ so that $q(D) f(x) = 0$. [This does not work for all $f(x)$.]
- Let $r_1, \ldots, r_n$ be the ("old") roots of the polynomial $p(D)$.
  Let $s_1, \ldots, s_m$ be the ("new") roots of the polynomial $q(D)$.
- It follows that y_p solves $q(D)\, p(D) y = 0$.
  The characteristic polynomial of this DE has roots $r_1, \ldots, r_n, s_1, \ldots, s_m$.
  Let $v_1, \ldots, v_m$ be the "new" solutions (i.e. not solutions of the "old" $p(D) y = 0$).
  By plugging into $p(D) y_p = f(x)$, we find (unique) C_i so that $y_p = C_1 v_1 + \ldots + C_m v_m$.

For which $f(x)$ does this work? By Theorem 25, we know exactly which $f(x)$ are solutions to homogeneous linear DEs with constant coefficients: these are linear combinations of exponentials $x^j e^{rx}$ (which includes $x^j e^{ax} \cos(bx)$ and $x^j e^{ax} \sin(bx)$).

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Example 33. Find the general solution of $y'' + 4y' + 4y = 7e^{-2x}$.

Solution. The "old" roots are $-2, -2$. The "new" roots are -2. Hence, there has to be a particular solution of the form $y_p = Cx^2 e^{-2x}$. To find the value of C, we plug into the DE.

$y_p' = C(-2x^2 + 2x)e^{-2x}$

$y_p'' = C(4x^2 - 8x + 2)e^{-2x}$

$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} \overset{!}{=} 7e^{-2x}$

It follows that $C = 7/2$, so that $y_p = \frac{7}{2}x^2 e^{-2x}$. The general solution is $y(x) = \left(C_1 + C_2 x + \frac{7}{2}x^2\right)e^{-2x}$.

Example 34. Find a particular solution of $y'' + 4y' + 4y = x\cos(x)$.

Solution. The "old" roots are $-2, -2$. The "new" roots are $\pm i, \pm i$. Hence, there has to be a particular solution of the form $y_p = (C_1 + C_2 x)\cos(x) + (C_3 + C_4 x)\sin(x)$. To find the value of the C_j's, we plug into the DE.

$y_p' = (C_2 + C_3 + C_4 x)\cos(x) + (C_4 - C_1 - C_2 x)\sin(x)$

$y_p'' = (2C_4 - C_1 - C_2 x)\cos(x) + (-2C_2 - C_3 - C_4 x)\sin(x)$

$y_p'' + 4y_p' + 4y_p = (3C_1 + 4C_2 + 4C_3 + 2C_4 + (3C_2 + 4C_4)x)\cos(x)$

$+ (-4C_1 - 2C_2 + 3C_3 + 4C_4 + (-4C_2 + 3C_4)x)\sin(x) \overset{!}{=} x\cos(x)$.

Equating the coefficients of $\cos(x)$, $x\cos(x)$, $\sin(x)$, $x\sin(x)$, we get the equations $3C_1 + 4C_2 + 4C_3 + 2C_4 = 0$, $3C_2 + 4C_4 = 1$, $-4C_1 - 2C_2 + 3C_3 + 4C_4 = 0$, $-4C_2 + 3C_4 = 0$.

Solving, we find $C_1 = -\frac{4}{125}$, $C_2 = \frac{3}{25}$, $C_3 = -\frac{22}{125}$, $C_4 = \frac{4}{25}$. [Make sure you know how to do this tedious step.]

Hence, $y_p = \left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)$.

Example 35. (extra) Find a particular solution of $y'' + 4y' + 4y = 5e^{-2x} - 3x\cos(x)$.

Solution. Instead of starting all over, recall that we already found $y_\triangle$ in Example 33 such that $Ly_\triangle = 7e^{-2x}$ (here, we write $L = p(D)$). Also, from Example 34 we have $y_\diamond$ such that $Ly_\diamond = x\cos(x)$.

By linearity, it follows that $L\left(\frac{5}{7}y_\triangle - 3y_\diamond\right) = \frac{5}{7}Ly_\triangle - 3Ly_\diamond = 5e^{-2x} - 3x\cos(x)$.

Hence, $y_p = \frac{5}{7}y_\triangle - 3y_\diamond = \frac{5}{2}x^2 e^{-2x} - 3\left[\left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)\right]$.

Example 36. (extra) Find a particular solution of $y'' + 4y' + 4y = 4e^{3x}\sin(2x) - x\sin(x)$.

Solution. The "old" roots are $-2, -2$. The "new" roots are $3 \pm 2i, \pm i, \pm i$.

Hence, there has to be a particular solution of the form

$y_p = C_1 e^{3x}\cos(2x) + C_2 e^{3x}\sin(2x) + (C_3 + C_4 x)\cos(x) + (C_5 + C_6 x)\sin(x)$.

To find the values of $C_1, \ldots, C_6$, we plug into the DE. But this final step is so boring that we stop here.

Computers (currently?) cannot afford to be as selective; mine obediently calculated:

$y_p = -\frac{4}{841}e^{3x}(20\cos(2x) - 21\sin(2x)) + \frac{1}{125}((-22 + 20x)\cos(x) + (4 - 15x)\sin(x))$

Example 37. We have been factoring differential operators like $D^2 + 4D + 4 = (D+2)^2$.

Things become much more complicated when the coefficients are not constant!

For instance, the linear DE $y'' + 4y' + 4xy = 0$ can be written as $Ly = 0$ with $L = D^2 + 4D + 4x$. However, in general, such operators cannot be factored (unless we allow as coefficients functions in x that we are not familiar with). [On the other hand, any ordinary polynomial can be factored over the complex numbers.]

One indication that things become much more complicated is that x and D do not commute: $xD \neq Dx$!!

Indeed, $(xD)f(x) = xf'(x)$ while $(Dx)f(x) = \frac{d}{dx}[xf(x)] = f(x) + xf'(x) = (1 + xD)f(x)$.

This computation shows that, in fact, $Dx = xD + 1$.

More next time!

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Review. Linear DEs are those that can be written as $Ly = f(x)$ where L is a linear differential operator: namely,

$$L = p_n(x)D^n + p_{n-1}(x)D^{n-1} + \ldots + p_1(x)D + p_0(x). \qquad (1)$$

Recall that the operators xD and Dx are not the same: instead, $Dx = xD + 1$.
We say that an operator of the form (1) is in **normal form**.

For instance. xD is in normal, whereas Dx is not in normal form. The normal form of Dx is $xD + 1$.

Example 38. Let $a = a(x)$ be some function.

(a) Write the operator Da in normal form [normal form means as in (1)].

(b) Write the operator $D^2 a$ in normal form.

Solution.

(a) $(Da)f(x) = \frac{\mathrm{d}}{\mathrm{d}x}[a(x)\, f(x)] = a'(x)f(x) + a(x)f'(x) = (a' + aD)f(x)$
Hence, $Da = aD + a'$.

(b) $(D^2 a)f(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}[a(x)\, f(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[a'(x)f(x) + a(x)f'(x)] = a''(x)f(x) + 2a'(x)f'(x) + a(x)f''(x)$
$= (a'' + 2a'D + aD^2)f(x)$
Hence, $D^2 a = aD^2 + 2a'D + a''$.

Example 39. Suppose that a and b depend on x. Expand $(D + a)(D + b)$ in normal form.

Solution. $(D + a)(D + b) = D^2 + Db + aD + ab = D^2 + (bD + b') + aD + ab = D^2 + (a + b)D + ab + b'$

Comment. Of course, if b is a constant, then $b' = 0$ and we just get the familiar expansion.
Comment. At this point, it is not surprising that, in general, $(D + a)(D + b) \neq (D + b)(D + a)$.

Example 40. Suppose we want to factor $D^2 + pD + q$ as $(D + a)(D + b)$. [p, q, a, b depend on x]

(a) Spell out equations to find a and b.

(b) Find all factorizations of D^2. [An obvious one is $D^2 = D \cdot D$ but there is others!]

Solution.

(a) Matching coefficients with $(D + a)(D + b) = D^2 + (a + b)D + ab + b'$, we find that we need

$$p = a + b, \quad q = ab + b'.$$

Equivalently, $a = p - b$ and $q = (p - b)b + b'$. The latter is a nonlinear (!) DE for b. Once solved for b, we obtain a as $a = p - b$.

(b) This is the case $p = q = 0$. The DE for b becomes $b' = b^2$.
Because it is separable (show all details!), we find that $b(x) = \frac{1}{C - x}$ or $b(x) = 0$.
Since $a = -b$, we obtain the factorizations $D^2 = \left(D - \frac{1}{C - x}\right)\left(D + \frac{1}{C - x}\right)$ and $D^2 = D \cdot D$.
Our computations show that there are no further factorizations.

Comment. Note that this example illustrates that factorization of differential operators is not unique!
For instance, $D^2 = D \cdot D$ and $D^2 = \left(D + \frac{1}{x}\right) \cdot \left(D - \frac{1}{x}\right)$ (the case $C = 0$ above).

Comment. In general, the nonlinear DE for b does not have any polynomial or rational solution (or, in fact, any solution that can be expressed in terms of functions that we are familiar with).

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A crash course in computing determinants

Review. The **determinant** of A, written as $\det(A)$ or $|A|$, is a number with the property that:

$$\begin{aligned}\det(A)\neq 0 &\iff A \text{ is invertible}\\ &\iff A\boldsymbol{x}=\boldsymbol{b} \text{ has a (unique) solution } \boldsymbol{x} \text{ (for all } \boldsymbol{b})\\ &\iff A\boldsymbol{x}=0 \text{ is only solved by } \boldsymbol{x}=0\end{aligned}$$

Example 41. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example 42. Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution. We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1\cdot\begin{vmatrix} + & & \\ & -1 & 2 \\ & 0 & 1 \end{vmatrix} - 2\cdot\begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + 0\cdot\begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix}$$

$$\overset{\text{i.e.}}{=} 1\cdot\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2\cdot\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0\cdot\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 1\cdot(-1) - 2\cdot(-1) + 0 = 1$$

Each term in the cofactor expansion is ± 1 times an entry times a smaller determinant (row and column of entry deleted).

The ± 1 is assigned to each entry according to $\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$.

Solution. We expand by the second column:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -2\cdot\begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + (-1)\cdot\begin{vmatrix} 1 & & 0 \\ & + & \\ 2 & & 1 \end{vmatrix} - 0\cdot\begin{vmatrix} 1 & & 0 \\ 3 & & 2 \\ & - & \end{vmatrix}$$

$$= -2\cdot(-1) + (-1)\cdot 1 - 0 = 1$$

Example 43. Compute $\begin{vmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 8 & 5 \end{vmatrix}$.

Solution. We can expand by the second column:

$$\begin{vmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 8 & 5 \end{vmatrix} = -0\begin{vmatrix} 0 & 1 & 5 \\ 0 & 2 & 1 \\ 2 & 8 & 5 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 2 & 8 & 5 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 2 & 8 & 5 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 2 & 1 \end{vmatrix}$$

[Of course, you don't have to spell out the 3×3 matrices that get multiplied with 0.]

We can compute the remaining 3×3 matrix in any way we prefer. One option is to expand by the first column:

$$2\begin{vmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 2 & 8 & 5 \end{vmatrix} = 2\left(+1\begin{vmatrix} 2 & 1 \\ 8 & 5 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix}\right) = 2(1\cdot 2 + 2\cdot(-5)) = -16$$

Comment. For cofactor expansion, choosing to expand by the second column is the best choice because this column has more zeros than any other column or row.

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Solving linear recurrences with constant coefficients

Motivation: Fibonacci numbers

The numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ are called **Fibonacci numbers**.

They are defined by the recursion $F_{n+1} = F_n + F_{n-1}$ and $F_0 = 0$, $F_1 = 1$.

How fast are they growing?

Have a look at ratios of Fibonacci numbers: $\frac{2}{1} = 2$, $\frac{3}{2} = 1.5$, $\frac{5}{3} \approx 1.667$, $\frac{8}{5} = 1.6$, $\frac{13}{8} = 1.625$, $\frac{21}{13} = 1.615$, $\frac{34}{21} = 1.619$, ...

These ratios approach the **golden ratio** $\varphi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$

In other words, it appears that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$. This indeed follows from Theorem 47 below.

We can derive all of that using the same ideas as in the case of linear differential equations. The crucial observation that we can write the recursion in operator form:

$F_{n+1} = F_n + F_{n-1}$ is equivalent to $(N^2 - N - 1)F_n = 0$.

Here, N is the shift operator: $Na_n = a_{n+1}$.

Comment. Recurrence equations are discrete analogs of differential equations.

For instance, recall that $f'(x) \approx f(x+1) - f(x)$ so that D is approximated by $N - 1$.

Example 44. Find the general solution to the recursion $a_{n+1} = 7a_n$.

Solution. Note that $a_n = 7a_{n-1} = 7 \cdot 7a_{n-2} = \cdots = 7^n a_0$.

Hence, the general solution is $a_n = C \cdot 7^n$.

Comment. This is analogous to $y' = 7y$ having the general solution $y(x) = Ce^{7x}$.

Example 45. Find the general solution to the recursion $a_{n+2} = a_{n+1} + 6a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 6 = (N-3)(N+2)$.

Since $(N-3)a_n = 0$ has solution $a_n = C \cdot 3^n$, and since $(N+2)a_n = 0$ has solution $a_n = C \cdot (-2)^n$ (compare previous example), we conclude that the general solution is $a_n = C_1 \cdot 3^n + C_2 \cdot (-2)^n$.

Comment. This must indeed be the general solution, because the two degrees of freedom C_1, C_2 allow us to match any initial conditions $a_0 = A$, $a_1 = B$: the two equations $C_1 + C_2 = A$ and $3C_1 - 2C_2 = B$ in matrix form are $\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$, which always has a (unique) solution because $\det\left(\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}\right) = -5 \neq 0$.

Example 46. Find the general solution to the recursion $a_{n+3} = 2a_{n+2} + a_{n+1} - 2a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^3 - 2N^2 - N + 2$ has roots $2, 1, -1$.

Hence, the general solution is $a_n = C_1 \cdot 2^n + C_2 + C_3 \cdot (-1)^n$.

Theorem 47. (Binet's formula) $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

Proof. The recursion $F_{n+1} = F_n + F_{n-1}$ can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 1$ has roots

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

Hence, $F_n = C_1 \cdot \lambda_1^n + C_2 \cdot \lambda_2^n$ and we only need to figure out the two unknowns C_1, C_2. We can do that using the two initial conditions: $F_0 = C_1 + C_2 \overset{!}{=} 0$, $F_1 = C_1 \cdot \frac{1+\sqrt{5}}{2} + C_2 \cdot \frac{1-\sqrt{5}}{2} \overset{!}{=} 1$.

Solving, we find $C_1 = \frac{1}{\sqrt{5}}$ and $C_2 = -\frac{1}{\sqrt{5}}$ so that, in conclusion, $F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$, as claimed. □

Comment. For large n, $F_n \approx \frac{1}{\sqrt{5}}\lambda_1^n$ (because λ_2^n becomes very small). In fact, $F_n = \operatorname{round}\left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$.

Back to the quotient of Fibonacci numbers. In particular, because λ_1^n dominates λ_2^n, it is now transparent that the ratios $\frac{F_{n+1}}{F_n}$ approach $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$. To be precise, note that

$$\frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}}(\lambda_1^{n+1} - \lambda_2^{n+1})}{\frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} = \frac{\lambda_1 - \lambda_2\left(\frac{\lambda_2}{\lambda_1}\right)^n}{1 - \left(\frac{\lambda_2}{\lambda_1}\right)^n} \overset{n\to\infty}{\longrightarrow} \frac{\lambda_1 - 0}{1 - 0} = \lambda_1.$$

In fact, it follows from $\lambda_2 < 0$ that the ratios $\frac{F_{n+1}}{F_n}$ approach λ_1 in the alternating fashion that we observed numerically earlier. Can you see that?

Example 48. Find the general solution to the recursion $a_{n+2} = 4a_{n+1} - 4a_n$.

Solution. The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - 4N + 4$ has roots $2, 2$.

So a solution is 2^n and, from our discussion of DEs, it is probably not surprising that a second solution is $n \cdot 2^n$.

Hence, the general solution is $a_n = C_1 \cdot 2^n + C_2 \cdot n \cdot 2^n = (C_1 + C_2 n) \cdot 2^n$.

Comment. This is analogous to $(D-2)^2 y' = 0$ having the general solution $y(x) = (C_1 + C_2 x)e^{2x}$.

Check! Let's check that $a_n = n \cdot 2^n$ indeed satisfies the recursion $(N-2)^2 a_n = 0$.

$(N-2)n \cdot 2^n = (n+1)2^{n+1} - 2n \cdot 2^n = 2^{n+1}$, so that $(N-2)^2 n \cdot 2^n = (N-2)2^{n+1} = 0$.

Combined, we obtain the following analog of Theorem 25 for recurrence equations (RE):

Solutions to such recurrences are called **C-finite sequences**.

Theorem 49. Consider the homogeneous linear RE with constant coefficients $p(N)a_n = 0$.

- If r is a root of the characteristic polynomial and if k is its multiplicity, then k (independent) solutions of the RE are given by $n^j r^n$ for $j = 0, 1, \ldots, k-1$.
- Combining these solutions for all roots, gives the general solution.

Moreover. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ equals the largest root r that contributes to a_n.

Example 50. (homework) Consider the sequence a_n defined by $a_{n+2} = 2a_{n+1} + 4a_n$ and $a_0 = 0$, $a_1 = 1$. Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

First few terms of sequence. $0, 1, 2, 8, 24, 80, 256, 832, \ldots$

These are actually related to Fibonacci numbers. Indeed, $a_n = 2^{n-1} F_n$. Can you prove this directly from the recursions? Alternatively, this follows from the Binet formulas.

Solution. Proceeding as for the Fibonacci numbers, we find $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{5} \approx 3.23607$.

Comment. With just a little more work, we find the Binet formula $a_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2\sqrt{5}}$.

Crash course: Eigenvalues and eigenvectors

If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ (and $\boldsymbol{x} \neq \boldsymbol{0}$), then $\boldsymbol{x}$ is an **eigenvector** of A with **eigenvalue** λ (just a number).

Note that for the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to make sense, A needs to be a square matrix (i.e. $n \times n$).

Key observation:

$$\begin{aligned} & A\boldsymbol{x} = \lambda\boldsymbol{x} \\ \Longleftrightarrow\ & A\boldsymbol{x} - \lambda\boldsymbol{x} = \boldsymbol{0} \\ \Longleftrightarrow\ & (A - \lambda I)\boldsymbol{x} = \boldsymbol{0} \end{aligned}$$

This homogeneous system has a nontrivial solution $\boldsymbol{x}$ if and only if $\det(A - \lambda I) = 0$.

To find eigenvectors and eigenvalues of A:

(a) First, find the eigenvalues λ by solving $\det(A - \lambda I) = 0$.

$\det(A - \lambda I)$ is a polynomial in λ, called the **characteristic polynomial** of A.

(b) Then, for each eigenvalue λ, find corresponding eigenvectors by solving $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$.

Example 51. Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

Solution. The characteristic polynomial is:

$\det(A - \lambda I) = \det\left(\begin{bmatrix} 8-\lambda & -10 \\ 5 & -7-\lambda \end{bmatrix}\right) = (8-\lambda)(-7-\lambda) + 50 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$

Hence, the eigenvalues are $\lambda = 3$ and $\lambda = -2$.

- To find an eigenvector for $\lambda = 3$, we need to solve $\begin{bmatrix} 5 & -10 \\ 5 & -10 \end{bmatrix}\boldsymbol{x} = \boldsymbol{0}$.
  Hence, $\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$.
- To find an eigenvector for $\lambda = -2$, we need to solve $\begin{bmatrix} 10 & -10 \\ 5 & -5 \end{bmatrix}\boldsymbol{x} = \boldsymbol{0}$.
  Hence, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.

Check! $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

On the other hand, a random other vector like $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an eigenvector: $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ -9 \end{bmatrix} \neq \lambda\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Example 52. (homework) Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -6 \\ 1 & -4 \end{bmatrix}$.

Solution. (final answer only) $\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$, and $\boldsymbol{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -1$.

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Back to linear recurrences

Example 53. (review) Consider the sequence a_n defined by $a_{n+2} = a_{n+1} + 2a_n$ and $a_0 = 1$, $a_1 = 8$.

(a) Determine the first few terms of the sequence.

(b) Find a Binet-like formula for a_n.

(c) Determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

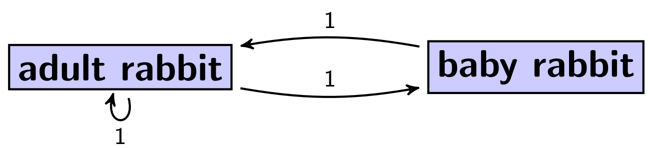
(a) $a_2 = 10$, $a_3 = 26$

(b) The recursion can be written as $p(N)a_n = 0$ where $p(N) = N^2 - N - 2$ has roots $2, -1$.
Hence, $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$ and we only need to figure out the two unknowns α_1, α_2. We can do that using the two initial conditions: $a_0 = \alpha_1 + \alpha_2 = 1$, $a_1 = 2\alpha_1 - \alpha_2 = 8$.
Solving, we find $\alpha_1 = 3$ and $\alpha_2 = -2$ so that, in conclusion, $a_n = 3 \cdot 2^n - 2(-1)^n$.

(c) It follows from the Binet-like formula that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$.

Example 54. We model rabbit reproduction as follows.

Each month, every pair of adult rabbits produces one pair of baby rabbit as offspring. Meanwhile, it takes baby rabbits one month to mature to adults.



Comment. In this simplified model, rabbits always come in male/female pairs and no rabbits die. Though these feautures might make it sound fairly useless, the model may have some merit when describing populations under ideal conditions (unlimited resources) and over short time (no deaths).

Historical comment. The question how many rabbits there are after one year, when starting out with a pair of baby rabbits is famously included in the 1202 textbook of the Italian mathematician Leonardo of Pisa, known as Fibonacci.

If we start with one baby rabbit pair, how many adult rabbits are there after n months?

Solution. Let a_n be the number of baby rabbit pairs after n months. Likewise, b_n is the number of adult rabbit pairs. The transition from one month to the next is given by $a_{n+1} = b_n$ and $b_{n+1} = a_n + b_n$. Using $a_n = b_{n-1}$ (that's an equivalent version of the first equation) in the second equation, we obtain $b_{n+1} = b_n + b_{n-1}$.

The initial conditions are $b_0 = 0$ and $b_1 = 0$.

It follows that the number b_n of adult rabbits are precisely the Fibonacci numbers F_n.

Comment. Note that the transition from one month to the next is described by in matrix-vector form as

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} b_n \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}.$$

Writing $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$, this becomes $\boldsymbol{a}_{n+1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \boldsymbol{a}_n$.

Consequently, $\boldsymbol{a}_n = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \boldsymbol{a}_0 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

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