Systems of linear DEs: the inhomogeneous case

Recall that any linear DE can be transformed into a first-order system. Hence, any linear DE (or any system of linear DEs) can written as

$$\boldsymbol{y}' = A(x)\,\boldsymbol{y} + \boldsymbol{f}(x).$$

Note. In general, A depends on x. In other words, the DE is allowed to have nonconstant coefficients.

Review. We showed in Theorem 17 that $y' = a(x)y + f(x)$ has the particular solution

$$y_p(x) = y_h(x)\int \frac{f(x)}{y_h(x)}\mathrm{d}x,$$

where $y_h(x) = e^{\int a(x)\mathrm{d}x}$ is any solution to the homogeneous equation $y' = a(x)y$.

Amazingly (or, maybe, by now, not surprisingly), the same arguments with the same result apply to systems of linear equations:

Theorem 71. (variation of constants) $\boldsymbol{y}' = A(x)\,\boldsymbol{y} + \boldsymbol{f}(x)$ has the particular solution

$$\boldsymbol{y}_p(x) = \Phi(x)\int \Phi(x)^{-1}\boldsymbol{f}(x)\mathrm{d}x,$$

where $\Phi(x)$ is any fundamental matrix solution to $\boldsymbol{y}' = A(x)\,\boldsymbol{y}$.

Proof. We can find this formula in the same manner as we did in Theorem 17:
Since the general solution of the homogeneous equation $\boldsymbol{y}' = A(x)\,\boldsymbol{y}$ is $\boldsymbol{y}_h = \Phi(x)\boldsymbol{c}$, we are going to vary the constant $\boldsymbol{c}$ and look for a particular solution of the form $\boldsymbol{y}_p = \Phi(x)\boldsymbol{c}(x)$. Plugging into the DE, we get:

$$\boldsymbol{y}_p' = \Phi'\boldsymbol{c} + \Phi\boldsymbol{c}' = A\Phi\boldsymbol{c} + \Phi\boldsymbol{c}' \overset{!}{=} A\boldsymbol{y}_p + \boldsymbol{f} = A\Phi\boldsymbol{c} + \boldsymbol{f}$$

For the first equality, we used the matrix version of the usual product rule (which holds since differentiation is defined entry-wise). For the second equality, we used $\Phi' = A\Phi$.
Hence, $\boldsymbol{y}_p = \Phi(x)\boldsymbol{c}(x)$ is a particular solution if and only if $\Phi\boldsymbol{c}' = \boldsymbol{f}$.
The latter condition means $\boldsymbol{c}' = \Phi^{-1}\boldsymbol{f}$ so that $\boldsymbol{c} = \int\Phi(x)^{-1}\boldsymbol{f}(x)\mathrm{d}x$, which gives the claimed formula for $\boldsymbol{y}_p$. □

Example 72. Find a particular solution to $\boldsymbol{y}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 0 \\ -2e^{3x} \end{bmatrix}$.

Solution. First, we determine (do it!) a fundamental matrix solution for $\boldsymbol{y}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\boldsymbol{y}$: $\Phi(x) = \begin{bmatrix} e^{-x} & 3e^{4x} \\ -e^{-x} & 2e^{4x} \end{bmatrix}$
Using $\det(\Phi(x)) = 5e^{3x}$, we find $\Phi(x)^{-1} = \frac{1}{5}\begin{bmatrix} 2e^{x} & -3e^{x} \\ e^{-4x} & e^{-4x} \end{bmatrix}$.

Hence, $\Phi(x)^{-1}\boldsymbol{f}(x) = \frac{2}{5}\begin{bmatrix} 3e^{4x} \\ -e^{-x} \end{bmatrix}$ and $\int\Phi(x)^{-1}\boldsymbol{f}(x)\mathrm{d}x = \frac{2}{5}\begin{bmatrix} 3/4e^{4x} \\ e^{-x} \end{bmatrix}$.

By variation of constants, $\boldsymbol{y}_p(x) = \Phi(x)\int\Phi(x)^{-1}\boldsymbol{f}(x)\mathrm{d}x = \begin{bmatrix} e^{-x} & 3e^{4x} \\ -e^{-x} & 2e^{4x} \end{bmatrix}\frac{2}{5}\begin{bmatrix} 3/4e^{4x} \\ e^{-x} \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}e^{3x}$.

In the special case that $\Phi(x) = e^{Ax}$, some things become easier. For instance, $\Phi(x)^{-1} = e^{-Ax}$. Also, we can just write down solutions to IVPs:

- $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$ has (unique) solution $\boldsymbol{y}(x) = e^{Ax}\boldsymbol{c}$.
- $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(x)$, $\boldsymbol{y}(0) = \boldsymbol{c}$ has (unique) solution $\boldsymbol{y}(x) = e^{Ax}\boldsymbol{c} + e^{Ax}\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t$.

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Example 73. Suppose that the matrix A satisfies $e^{Ax} = \begin{bmatrix} 2e^{2x} - e^{3x} & -2e^{2x} + 2e^{3x} \\ e^{2x} - e^{3x} & -e^{2x} + 2e^{3x} \end{bmatrix}$.

(a) Solve $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) Solve $\boldsymbol{y}' = A\boldsymbol{y} + \begin{bmatrix} 0 \\ 2e^x \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) What is A?

Solution.

(a) $\boldsymbol{y}(x) = e^{Ax}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2e^{2x} + 3e^{3x} \\ -e^{2x} + 3e^{3x} \end{bmatrix}$

(b) $\boldsymbol{y}(x) = e^{Ax}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{Ax}\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t$. We compute:

$\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t = \int_0^x \begin{bmatrix} 2e^{-2t} - e^{-3t} & -2e^{-2t} + 2e^{-3t} \\ e^{-2t} - e^{-3t} & -e^{-2t} + 2e^{-3t} \end{bmatrix}\begin{bmatrix} 0 \\ 2e^t \end{bmatrix}\mathrm{d}t = \int_0^x \begin{bmatrix} -4e^{-t} + 4e^{-2t} \\ -2e^{-t} + 4e^{-2t} \end{bmatrix}\mathrm{d}t = \begin{bmatrix} 4e^{-x} - 2e^{-2x} - 2 \\ 2e^{-x} - 2e^{-2x} \end{bmatrix}$

Hence, $e^{Ax}\int_0^x e^{-At}\boldsymbol{f}(t)\mathrm{d}t = \begin{bmatrix} 2e^{2x} - e^{3x} & -2e^{2x} + 2e^{3x} \\ e^{2x} - e^{3x} & -e^{2x} + 2e^{3x} \end{bmatrix}\begin{bmatrix} 4e^{-x} - 2e^{-2x} - 2 \\ 2e^{-x} - 2e^{-2x} \end{bmatrix} = \begin{bmatrix} 2e^x - 4e^{2x} + 2e^{3x} \\ -2e^{2x} + 2e^{3x} \end{bmatrix}$.

Finally, $\boldsymbol{y}(x) = \begin{bmatrix} -2e^{2x} + 3e^{3x} \\ -e^{2x} + 3e^{3x} \end{bmatrix} + \begin{bmatrix} 2e^x - 4e^{2x} + 2e^{3x} \\ -2e^{2x} + 2e^{3x} \end{bmatrix} = \begin{bmatrix} 2e^x - 6e^{2x} + 5e^{3x} \\ -3e^{2x} + 5e^{3x} \end{bmatrix}$.

(c) Like any fundamental matrix, $\Phi = e^{Ax}$ satisfies $\frac{\mathrm{d}}{\mathrm{d}x}e^{Ax} = Ae^{Ax}$.

Hence, $A = \left[\frac{\mathrm{d}}{\mathrm{d}x}e^{Ax}\right]_{x=0} = \left[\begin{bmatrix} 4e^{2x} - 3e^{3x} & -4e^{2x} + 6e^{3x} \\ 2e^{2x} - 3e^{3x} & -2e^{2x} + 6e^{3x} \end{bmatrix}\right]_{x=0} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

Modeling

Example 74. Consider two brine tanks. Tank T_1 contains 24gal water containing 3lb salt, and tank T_2 contains 9gal pure water.

- T_1 is being filled with 54gal/min water containing 0.5lb/gal salt.
- 72gal/min well-mixed solution flows out of T_1 into T_2.
- 18gal/min well-mixed solution flows out of T_2 into T_1.
- Finally, 54gal/min well-mixed solution is leaving T_2.

How much salt is in the tanks after t minutes?

Solution. Note that the amount of water in each tank is constant because the flows balance each other. Let $y_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min). In time interval $[t, t + \Delta t]$:

$\Delta y_1 \approx 54 \cdot \frac{1}{2} \cdot \Delta t - 72 \cdot \frac{y_1}{24} \cdot \Delta t + 18 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_1' = 27 - 3y_1 + 2y_2$. Also, $y_1(0) = 3$.

$\Delta y_2 \approx 72 \cdot \frac{y_1}{24} \cdot \Delta t - 72 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_2' = 3y_1 - 8y_2$. Also, $y_2(0) = 0$.

Using matrix notation and writing $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this is $\boldsymbol{y}' = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

This is an IVP that we can solve (with some work)! Do it! Skipping most work, we find:

- $e^{At} = \frac{1}{7}\begin{bmatrix} e^{-9t} + 6e^{-2t} & -2e^{-9t} + 2e^{-2t} \\ -3e^{-9t} + 3e^{-2t} & 6e^{-9t} + 1e^{-2t} \end{bmatrix}$
- $\boldsymbol{y} = e^{At}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{At}\int_0^t e^{-As}\begin{bmatrix} 27 \\ 0 \end{bmatrix}\mathrm{d}s = \frac{1}{7}\begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -3e^{-9t} + 3e^{-2t} \end{bmatrix} + \frac{3}{14}e^{At}\begin{bmatrix} 2e^{9t} + 54e^{2t} - 56 \\ -6e^{9t} + 27e^{2t} - 21 \end{bmatrix} = \begin{bmatrix} 12 - 9e^{-2t} \\ 4.5 - 4.5e^{-2t} \end{bmatrix}$

Note. We could have found a particular solution with less calculations by observing (looking at "old" and "new" roots) that there must be a solution of the form $\boldsymbol{y}_p(t) = \boldsymbol{a}$. We can then find $\boldsymbol{a}$ by plugging into the differential equation. However, noticing that, for a constant solution, each tank has to have a constant concentration of 3lb/gal of salt, we find $\boldsymbol{y}_p(t) = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$.

Two more applications of systems of DEs

Example 75. (epidemiology) Let us indicate the popular SIR model for short outbreaks of diseases among a population of constant size N.

In a SIR model, the population is compartmentalized into $S(t)$ susceptible, $I(t)$ infected and $R(t)$ recovered (or resistant) individuals ($N = S(t) + I(t) + R(t)$). In the Kermack-McKendrick model, the outbreak of a disease is modeled by

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I,$$

with γ modeling the recovery rate and β the infection rate. Note that this is a non-linear system of differential equations. For more details and many variations used in epidemiology, see:

https://en.wikipedia.org/wiki/Compartmental_models_in_epidemiology

Comment. The following variation

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I R, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I R,$$

which assumes "infectious recovery", was used in 2014 to predict that facebook might lose 80% of its users by 2017. It's that claim, not mathematics (or even the modeling), which attracted a lot of media attention.

http://blogs.wsj.com/digits/2014/01/22/controversial-paper-predicts-facebook-decline/

Example 76. (military strategy) Lanchester's equations model two opposing forces during "aimed fire" battle.

Let $x(t)$ and $y(t)$ describe the number of troops on each side. Then Lanchester (during World War I) assumed that the rates $-x'(t)$ and $-y'(t)$, at which soldiers are put out of action, are proportional to the number of opposing forces. That is:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\alpha y(t) \\ -\beta x(t) \end{bmatrix}, \quad \text{or, in matrix form: } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \\ -\beta & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The proportionality constants $\alpha, \beta > 0$ indicate the strength of the forces ("fighting effectiveness coefficients").

These are simple linear DEs with constant coefficients, which we have learned how to solve.

For more details, see: https://en.wikipedia.org/wiki/Lanchester%27s_laws

Comment. The "aimed fire" means that all combatants are engaged, as is common in modern combat with long-range weapons. This is rather different than ancient combat where soldiers were engaging one opponent at a time.

Some special functions and the power series method

Review: power series

Definition 77. A function $y(x)$ is analytic around $x = x_0$ if it has a power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Note. In the next theorem, we will see that this power series is the Taylor series of $y(x)$ around $x = x_0$.

Power series are very pleasant to work with because they behave just like polynomials. For instance, we can differentiate and integrate them:

- If $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then $y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$ (another power series!).

  Note that $y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n$. Likewise, for higher derivatives.

- $\int y(x)\mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} + C$

Theorem 78. If $y(x)$ is analytic around $x = x_0$, then $y(x)$ is infinitely differentiable and

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{with} \quad a_n = \frac{y^{(n)}(x_0)}{n!}.$$

Caution. Analyticity is needed in this theorem; being infinitely differentiable is not enough. For instance, $y(x) = e^{-1/x^2}$ is infinitely differentiable around $x = 0$ (and everywhere else). However, $y^{(n)}(0) = 0$ for all n.

We have already seen the following example.

Example 79. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots$

Once again, notice how the power series clearly has the property that $y' = y$.

It follows from here that, for instance, $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \ldots$

Example 80. Determine a power series for $\cos(x)$.

Solution. (via DE) $\cos(x)$ is the unique solution to the IVP $y'' = -y$, $y(0) = 1$, $y'(0) = 0$.

It follows that $\cos(x) = \sum_{n=0}^{\infty} a_n x^n$ with $a_n = \frac{y^{(n)}(0)}{n!}$. The DE implies that $y^{(2n)}(x) = (-1)^n y(x)$ and $y^{(2n+1)} = (-1)^n y'(x)$ so that $y^{(2n)}(0) = (-1)^n$ and $y^{(2n+1)}(0) = 0$. Consequently, $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.

Solution. (via Euler's formula) Recall that $e^{ix} = \cos(x) + i\sin(x)$. Since

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{m=0}^{\infty} \frac{(ix)^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(ix)^{2m+1}}{(2m+1)!} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!},$$

we conclude that $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ and $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$.

Example 81. (Airy equation, to be cont'd) Let $y(x)$ be the unique solution to the IVP $y'' = xy$, $y(0) = a$, $y'(0) = b$. Determine the first several terms (up to x^6) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0) = 0 \cdot y(0) = 0$.

Differentiating both sides of the DE, we obtain $y''' = y + xy'$ so that $y'''(0) = y(0) + 0 \cdot y'(0) = a$.

Likewise, $y^{(4)} = 2y' + xy''$ shows $y^{(4)}(0) = 2y'(0) = 2b$.

Continuing, $y^{(5)} = 3y'' + xy'''$ so that $y^{(5)}(0) = 3y''(0) = 0$.

Continuing, $y^{(6)} = 4y''' + xy^{(4)}$ so that $y^{(6)}(0) = 4y'''(0) = 4a$.

Hence, $y(x) = a + bx + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{24}y^{(4)}(0)x^4 + \frac{1}{120}y^{(5)}(0)x^5 + \frac{1}{720}y^{(6)}(0)x^6 + \ldots$
$= a + bx + \frac{a}{6}x^3 + \frac{b}{12}x^4 + \frac{a}{180}x^6 + \ldots$

Comment. Do you see the general pattern? We will revisit this example soon.

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Notation. When working with power series $\sum_{n=0}^{\infty} a_n x^n$, we sometimes write $O(x^n)$ to indicate that we omit terms that are multiples of x^n:

For instance. $e^x = 1 + x + \frac{1}{2}x^2 + O(x^3)$ or $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O(x^6)$.

Power series solutions to DE

Given any DE, we can approximate analytic solutions by working with the first few terms of the power series.

Example 82. Let $y(x)$ be the unique solution to the IVP $y' = x^2 + y^2$, $y(0) = 1$.

Determine the first several terms (up to x^4) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y'(0) = 0^2 + y(0)^2 = 1$.

Differentiating both sides of the DE, we obtain $y'' = 2x + 2yy'$. In particular, $y''(0) = 2$.

Continuing, $y''' = 2 + 2(y')^2 + 2yy''$ so that $y'''(0) = 2 + 2 + 2 \cdot 2 = 8$.

Likewise, $y^{(4)} = 6y'y'' + 2yy'''$ so that $y^{(4)}(0) = 12 + 16 = 28$.

Hence, $y(x) = y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{24}y^{(4)}(0)x^4 + \ldots = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \ldots$

Comment. This approach requires the (symbolic) computation of intermediate derivatives. This is costly (even just the size of the simplified formulas is quickly increasing) and so the solution below is usually preferable for practical purposes. However, successive differentiation works well when working by hand.

Solution. (plug in power series) The powers series $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ simplifies to $y = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ because of the initial condition.

Therefore, $y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$

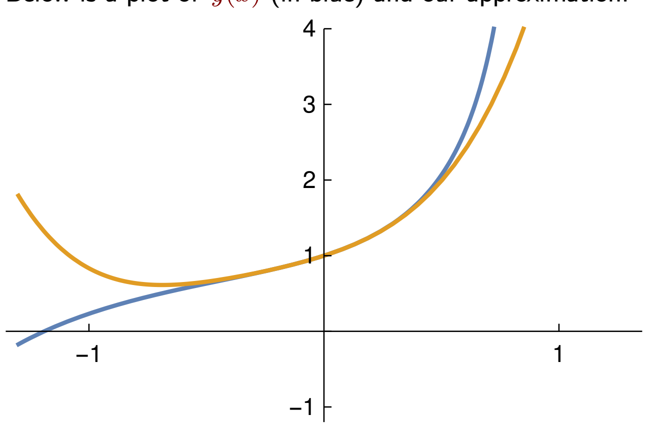
To determine a_2, a_3, a_4, a_5, we need to expand $x^2 + y^2$ into a power series:

$$y^2 = 1 + 2a_1x + (2a_2 + a_1^2)x^2 + (2a_3 + 2a_1a_2)x^3 + (2a_4 + 2a_1a_3 + a_2^2)x^4 + \ldots$$ [we don't need the last term]

Equating coefficients of y' and $x^2 + y^2$, we find $a_1 = 1$, $2a_2 = 2a_1$, $3a_3 = 1 + 2a_2 + a_1^2$, $4a_4 = 2a_3 + 2a_1a_2$.

So $a_1 = 1$, $a_2 = 1$, $a_3 = \frac{4}{3}$, $a_4 = \frac{7}{6}$ and, hence, $y(x) = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \ldots$

Below is a plot of $y(x)$ (in blue) and our approximation:



Note how the approximation is very good close to 0 but does not provide us with a "global picture".

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Example 83. Let $y(x)$ be the unique solution to the IVP $y'' = \cos(x+y)$, $y(0)=0$, $y'(0)=1$. Determine the first several terms (up to x^5) in the power series of $y(x)$.

Solution. (successive differentiation) From the DE, $y''(0) = \cos(0+y(0)) = 1$.

Differentiating both sides of the DE, we obtain $y''' = -\sin(x+y)(1+y')$.

In particular, $y'''(0) = -\sin(0+y(0))(1+y'(0)) = 0$.

Likewise, $y^{(4)} = -\cos(x+y)(1+y')^2 - \sin(x+y)y''$ shows $y^{(4)}(0) = -1\cdot 2^2 - 0 = -4$.

Continuing, $y^{(5)} = \sin(x+y)(1+y')^3 - 3\cos(x+y)(1+y')y'' - \sin(x+y)y'''$ so that $y^{(5)}(0) = -6$.

Hence, $y(x) = x + \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{1}{24}y^{(4)}(0)x^4 + \frac{1}{120}y^{(5)}(0)x^5 + \ldots = x + \frac{1}{2}x^2 - \frac{1}{6}x^4 - \frac{1}{20}x^5 + \ldots$

Solution. (plug in power series) The powers series $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$ simplifies to $y = x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$ because of the initial conditions.

Therefore, $y' = 1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$ and $y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \ldots$

To determine a_2, a_3, a_4, a_5, we need to expand $\cos(x+y)$ into a power series:

Recall that $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots$

Hence, $\cos(x+y) = 1 - \frac{1}{2}(x+y)^2 + \frac{1}{24}(x+y)^4 + \ldots = 1 - \frac{1}{2}x^2 - xy - \frac{1}{2}y^2 + O(x^4)$.

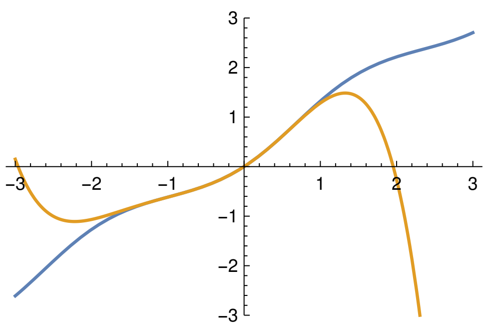
Since $y^2 = (x + a_2 x^2 + a_3 x^3 + \ldots)^2 = x^2 + 2a_2 x^3 + O(x^4)$,

$\cos(x+y) = 1 - \frac{1}{2}x^2 - x(x + a_2 x^2) - \frac{1}{2}(x^2 + 2a_2 x^3) + O(x^4) = 1 - 2x^2 - 2a_2 x^3 + O(x^4)$.

Equating coefficients of y'' and $\cos(x+y)$, we find $2a_2 = 1$, $6a_3 = 0$, $12a_4 = -2$, $20a_5 = -2a_2$.

So $a_2 = \frac{1}{2}$, $a_3 = 0$, $a_4 = -\frac{1}{6}$, $a_5 = -\frac{1}{20}$ and, hence, $y(x) = x + \frac{1}{2}x^2 - \frac{1}{6}x^4 - \frac{1}{20}x^5 + \ldots$

Below is a plot of $y(x)$ (in blue) and our approximation:



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Power series solutions to linear DEs

Note how in the last two examples the "plug in power series" approach was complicated by the fact that the DE was not linear (we had to expand y^2 as well as $\cos(x+y)$, respectively).

For linear DEs, this complication does not arise and we can readily determine the complete power series expansion of analytic solutions (with a recursive description of the coefficients).

Example 84. (Airy equation, cont'd) Let $y(x)$ be the unique solution to the IVP $y'' = xy$, $y(0) = a$, $y'(0) = b$. Determine the power series of $y(x)$.

Solution. (plug in power series) Let us spell out the power series for y, y', y'' and xy:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$xy(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Hence, $y'' = xy$ becomes $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = \sum_{n=1}^{\infty} a_{n-1} x^n$. We compare coefficients of x^n:

- $n = 0$: $2 \cdot 1 a_2 = 0$, so that $a_2 = 0$.
- $n \geqslant 1$: $(n+2)(n+1) a_{n+2} = a_{n-1}$

In conclusion, $y(x) = \sum_{n=0}^{\infty} a_n x^n$ with $a_0 = a$, $a_1 = b$, $a_2 = 0$ as well as, for $n \geqslant 3$, $a_n = \frac{1}{n(n-1)} a_{n-3}$.

First few terms. In particular, $y = a\left(1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{(2 \cdot 3)(5 \cdot 6)} + \ldots\right) + b\left(x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{(3 \cdot 4)(6 \cdot 7)} + \ldots\right)$.

Remark. The solution with $y(0) = \frac{1}{3^{2/3}\Gamma(2/3)}$ and $y'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)}$ is known as the **Airy function** $\mathrm{Ai}(x)$.
[A more natural property of $\mathrm{Ai}(x)$ is that it satisfies $y(x) \to 0$ as $x \to \infty$.]

Once we have a power series solution $y(x)$, a natural question is: for which x does the series converge?

Recall. A power series $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ has a **radius of convergence** R.
The series converges for all x with $|x - x_0| < R$ and it diverges for all x with $|x - x_0| > R$.

Definition 85. Consider the linear DE $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = f(x)$.
x_0 is called an **ordinary point** if the coefficients $p_j(x)$, as well as $f(x)$, are analytic at $x = x_0$.
Otherwise, x_0 is called a **singular point**.

Example 86. Determine the singular points of $(x+2) y'' - x^2 y + 3y = 0$.

Solution. Rewriting the DE as $y'' - \frac{x^2}{x+2} y + \frac{3}{x+2} y = 0$, we see that the only singular point is $x = -2$.

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Example 87. Determine the singular points of $(x^2+1)y''' = \frac{y}{x-5}$.

Solution. Rewriting the DE as $y''' - \frac{1}{(x-5)(x^2+1)} y = 0$, we see that the singular points are $x = \pm i, 5$.

Theorem 88. Consider the linear DE $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = f(x)$. Suppose that x_0 is an ordinary point and that R is the distance to the closest singular point. Then any IVP specifying $y(x_0)$, $y'(x_0)$, ..., $y^{(n-1)}(x_0)$ has a power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ and that series has radius of convergence at least R.

In particular. The DE has a general solution consisting of n solutions $y(x)$ that are analytic at $x = x_0$.

Comment. Most textbooks only discuss the case of 2nd order DEs. For a discussion of the higher order case (in terms of first order systems!) see, for instance, Chapter 4.5 in *Ordinary Differential Equations* by N. Lebovitz. The book is freely available at: `http://people.cs.uchicago.edu/~lebovitz/odes.html`

Example 89. Find a minimum value for the radius of convergence of a power series solution to $(x+2)y'' - x^2 y' + 3y = 0$ at $x = 3$.

Solution. As before, rewriting the DE as $y'' - \frac{x^2}{x+2} y' + \frac{3}{x+2} y = 0$, we see that the only singular point is $x = -2$.
Note that $x = 3$ is an ordinary point of the DE and that the distance to the singular point is $|3 - (-2)| = 5$.
Hence, the DE has power series solutions about $x = 3$ with radius of convergence at least 5.

Example 90. Find a minimum value for the radius of convergence of a power series solution to $(x^2+1)y''' = \frac{y}{x-5}$ at $x = 2$.

Solution. As before, rewriting the DE as $y''' - \frac{1}{(x-5)(x^2+1)} y = 0$, we see that the singular points are $x = \pm i, 5$.
Note that $x = 2$ is an ordinary point of the DE and that the distance to the nearest singular point is $|2 - i| = \sqrt{5}$ (the distances are $|2-5| = 3$, $|2 - i| = |2 - (-i)| = \sqrt{2^2 + 1^2} = \sqrt{5}$).
Hence, the DE has power series solutions about $x = 2$ with radius of convergence at least $\sqrt{5}$.

Example 91. (Airy equation, once more) Let $y(x)$ be the solution to the IVP $y'' = xy$, $y(0) = a$, $y'(0) = b$. Earlier, we determined the power series of $y(x)$. What is its radius of convergence?

Solution. $y'' = xy$ has no singular points. Hence, the radius of convergence is ∞. (In other words, the power series converges for all x.)