Midterm #1 – Practice

MATH 332 — Differential Equations II
Midterm: Thursday, Oct 1

Please print your name:

Problem 1.

(a) Find the general solution to $y^{(5)} - 4y^{(4)} + 5y''' - 2y'' = 0$.

(b) Find the general solution to $y''' - y = e^x + 7$.

(c) Solve $y'' + 2y' + y = 2e^{2x} + e^{-x}$, $y(0) = -1$, $y'(0) = 2$.

(d) Find the general solution to $y'' - 4y' + 4y = 3e^{2x}$.

(e) Consider a homogeneous linear differential equation with constant real coefficients which has order 6. Suppose $y(x) = x^2 e^{2x} \cos(x)$ is a solution. Write down the general solution.

(f) Write down a homogeneous linear differential equation satisfied by $y(x) = 1 - 5x^2 e^{-2x}$.

(g) Let y_p be any solution to the inhomogeneous linear differential equation $y'' + x y = e^x$. Find a homogeneous linear differential equation which y_p solves. *Hint:* Do not attempt to solve the DE.

Solution.

(a) The characteristic polynomial $p(D) = D^5 - 4D^4 + 5D^3 - 2D^2 = D^2(D-1)^2(D-2)$ has roots $0, 0, 1, 1, 2$.

Hence, the general solution is $y(x) = c_1 + c_2 x + (c_3 + c_4 x)e^x + c_5 e^{2x}$.

(b) The characteristic polynomial $p(D) = D^3 - 1$ of the associated homogeneous DE has roots 1 and $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. These are the "old" roots.

The "new" roots coming from $e^x + 7$ are $0, 1$. Hence, there has to be a particular solution of the form $y_p = A x e^x + B$. To find the values of A, B, we plug into the DE.

$y_p' = A(x+1)e^x$, $y_p'' = A(x+2)e^x$, $y_p''' = A(x+3)e^x$

$y_p''' - y_p = 3Ae^x - B \overset{!}{=} e^x + 7$

Consequently, $A = \frac{1}{3}$, $B = -7$.

Hence, the general solution is $y(x) = -7 + (c_1 + \frac{1}{3}x)e^x + c_2 e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$.

(c) The characteristic polynomial $p(D) = D^2 + 2D + 1$ of the associated homogeneous DE has roots $-1, -1$. These are the "old" roots.

The "new" roots coming from $2e^{2x} + e^{-x}$ are $-1, 2$. Hence, there has to be a particular solution of the form $y_p = Ae^{2x} + Bx^2 e^{-x}$. To find the values of A, B, we plug into the DE.

$y_p' = 2Ae^{2x} + B(2x - x^2)e^{-x}$, $y_p'' = 4Ae^{2x} + B(2 - 4x + x^2)e^{-x}$

$y_p'' + 2y_p' + y_p = 9Ae^{2x} + 2Be^{-x} \overset{!}{=} 2e^{2x} + e^{-x}$

Consequently, $A = \frac{2}{9}$, $B = \frac{1}{2}$.

Hence, the general solution is $y(x) = \frac{2}{9}e^{2x} + \frac{1}{2}x^2 e^{-x} + c_1 e^{-x} + c_2 x e^{-x}$. Now, we use the initial values, to find the values for c_1 and c_2:

$y(0) = \frac{2}{9} + c_1 \overset{!}{=} -1$, so that $c_1 = -\frac{11}{9}$.

Armin Straub
straub@southalabama.edu

$y'(0) = \left[\frac{4}{9}e^{2x} + \left(x - \frac{1}{2}x^2\right)e^{-x} + \frac{11}{9}e^{-x} + c_2(1-x)e^{-x}\right]_{x=0} = \frac{5}{3} + c_2 \overset{!}{=} 2$, so that $c_2 = \frac{1}{3}$.

In conclusion, the unique solution to the IVP is $y(x) = \frac{2}{9}e^{2x} + \frac{1}{2}x^2 e^{-x} - \frac{11}{9}e^{-x} + \frac{1}{3}x\,e^{-x}$.

(d) The characteristic polynomial $p(D) = D^2 - 4D - 4$ of the associated homogeneous DE has "old" roots $2, 2$.

The "new" roots coming from $3e^{2x}$ are 2. Hence, there has to be a particular solution of the form $y_p = Ax^2e^{2x}$. To find the value of A, we plug into the DE.

$y_p' = 2A(x + x^2)e^{2x}, \quad y_p'' = 2A(1 + 4x + 2x^2)e^{2x}$

$y_p'' - 4y_p' + 4y_p = [2A(1 + 4x + 2x^2) - 8A(x + x^2) + 4Ax^2]e^{2x} = 2Ae^{2x} \overset{!}{=} 3e^{2x}$. Consequently, $A = \frac{3}{2}$.

Hence, the general solution is $\left(c_1 + c_2\,x + \frac{3}{2}x^2\right)e^{2x}$.

(e) $y(x) = x^2e^{2x}\cos(x)$ is a solution of $p(D)y = 0$ if and only if $2 \pm i$ are three times repeated roots of the characteristic polynomial $p(D)$. Since the order of the DE is 6, there can be no further roots.

The general solution of this DE is $y(x) = (c_1 + c_2x + c_3x^2)e^{2x}\cos(x) + (c_4 + c_5x + c_6x^2)e^{2x}\sin(x)$.

(f) $y(x) = 1 - 5x^2e^{-2x}$ is a solution of $p(D)y = 0$ if and only if $-2, -2, -2, 0$ are roots of the characteristic polynomial $p(D)$. Hence, the simplest DE is obtained from $p(D) = D(D+2)^3 = D^4 + 6D^3 + 12D^2 + 8D$.

The corresponding recurrence is $y^{(4)} + 6y''' + 12y'' + 8y' = 0$.

(g) To kill e^x, we apply $D - 1$ to both sides of the DE $y'' + x\,y = e^x$.

The result is the homogeneous linear DE $y''' - y'' + xy' + (1-x)y = 0$.

Comment. If we are comfortable computing with operators, we can apply the relation $Dx = x\,D + 1$, to $(D-1)(D^2 + x) = D^3 - D^2 + Dx - x = D^3 - D^2 + x\,D + 1 - x$ to reach the same conclusion.

Problem 2.

(a) Write down a (homogeneous linear) recurrence equation satisfied by $a_n = 3^n - 2^n$.

(b) Write down a (homogeneous linear) recurrence equation satisfied by $a_n = n^2 3^n - 2^n$.

Solution.

(a) $a_n = 3^n - 2^n$ is a solution of $p(N)a_n = 0$ if and only if both 3 and 2 are a root of the characteristic polynomial $p(N)$. Hence, the simplest recurrence is obtained from $p(N) = (N-2)(N-3) = N^2 - 5N + 6$.

The corresponding recurrence is $a_{n+2} = 5a_{n+1} - 6a_n$.

(b) $a_n = n^2 3^n - 2^n$ is a solution of $p(N)a_n = 0$ if and only if 3 (repeated three times) and 2 are a root of the characteristic polynomial $p(N)$. Hence, the simplest recurrence is obtained from $p(N) = (N-2)(N-3)^3$.

The corresponding recurrence is $(N-2)(N-3)^3 a_n = 0$.

[Spelled out, this is $a_{n+4} = 11a_{n+3} - 45a_{n+2} + 81a_{n+1} - 54a_n$.]

Problem 3. Consider the sequence a_n defined by $a_{n+2} = a_{n+1} + 6a_n$ and $a_0 = 3$, $a_1 = -1$.

(a) Determine the first few terms of the sequence.

(b) Find a Binet-like formula for a_n.

(c) Determine $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $a_2 = 17$, $a_3 = 11$

(b) The recursion can be written as $p(N) a_n = 0$ where $p(N) = N^2 - N - 6$ has roots $3, -2$.

Hence, $a_n = \alpha_1 3^n + \alpha_2 (-2)^n$ and we only need to figure out the two unknowns α_1, α_2. We can do that using the two initial conditions: $a_0 = \alpha_1 + \alpha_2 = 3$, $a_1 = 3\alpha_1 - 2\alpha_2 = -1$.

Solving, we find $\alpha_1 = 1$ and $\alpha_2 = 2$ so that, in conclusion, $a_n = 3^n + 2 \cdot (-2)^n$.

(c) It follows from the Binet-like formula that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 3$.

Problem 4. Let $M = \left[\begin{smallmatrix} 1 & 4 \\ 6 & -1 \end{smallmatrix}\right]$.

(a) Determine the general solution to $\boldsymbol{a}_{n+1} = M \boldsymbol{a}_n$.

(b) Determine a fundamental matrix solution to $\boldsymbol{a}_{n+1} = M \boldsymbol{a}_n$.

(c) Compute M^n.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M - \lambda I) = \det\left(\left[\begin{smallmatrix} 1-\lambda & 4 \\ 6 & -1-\lambda \end{smallmatrix}\right]\right) = (1-\lambda)(-1-\lambda) - 24 = \lambda^2 - 25 = (\lambda - 5)(\lambda + 5)$

Hence, the eigenvalues are $\lambda = 5$ and $\lambda = -5$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = 5$, we need to solve $\left[\begin{smallmatrix} -4 & 4 \\ 6 & -6 \end{smallmatrix}\right] \boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ is an eigenvector for $\lambda = 5$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = -5$, we need to solve $\left[\begin{smallmatrix} 6 & 4 \\ 6 & 4 \end{smallmatrix}\right] \boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \left[\begin{smallmatrix} -2 \\ 3 \end{smallmatrix}\right]$ is an eigenvector for $\lambda = -5$.

Hence, the general solution is $C_1 \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] 5^n + C_2 \left[\begin{smallmatrix} -2 \\ 3 \end{smallmatrix}\right] (-5)^n$.

(b) The corresponding fundamental matrix solution is $\Phi_n = \left[\begin{smallmatrix} 5^n & -2(-5)^n \\ 5^n & 3(-5)^n \end{smallmatrix}\right]$.

(c) Note that $\Phi_0 = \left[\begin{smallmatrix} 1 & -2 \\ 1 & 3 \end{smallmatrix}\right]$, so that $\Phi_0^{-1} = \frac{1}{5}\left[\begin{smallmatrix} 3 & 2 \\ -1 & 1 \end{smallmatrix}\right]$. It follows that

$$M^n = \Phi_n \Phi_0^{-1} = \begin{bmatrix} 5^n & -2(-5)^n \\ 5^n & 3(-5)^n \end{bmatrix} \frac{1}{5} \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 \cdot 5^n + 2(-5)^n & 2 \cdot 5^n - 2(-5)^n \\ 3 \cdot 5^n - 3(-5)^n & 2 \cdot 5^n + 3(-5)^n \end{bmatrix}.$$

Problem 5.

(a) Write the differential equation $y''' + 7y'' - 3y' + y = 0$ as a system of (first-order) differential equations.

(b) Consider the following system of initial value problems:

$$\begin{aligned} y_1'' &= 3y_1' + 2y_2' - 5y_1 \\ y_2'' &= y_1' - y_2' + 3y_2 \end{aligned} \qquad y_1(0) = 1, \; y_1'(0) = -2, \; y_2(0) = 3, \; y_2'(0) = 0$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution.

(a) Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' + 7y'' - 3y' + y = 0$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -y_1 + 3y_2 - 7y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & -7 \end{bmatrix} \boldsymbol{y}$.

(b) Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 0 & 3 & 2 \\ 0 & 3 & 1 & -1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \end{bmatrix}.$$

Problem 6. Let $M = \begin{bmatrix} 11 & -2 \\ 3 & 4 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M - \lambda I) = \det\left(\begin{bmatrix} 11-\lambda & -2 \\ 3 & 4-\lambda \end{bmatrix}\right) = (11-\lambda)(4-\lambda) + 6 = \lambda^2 - 15\lambda + 50 = (\lambda - 5)(\lambda - 10)$

Hence, the eigenvalues are $\lambda = 5$ and $\lambda = 10$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = 5$, we need to solve $\begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector for $\lambda = 5$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = 10$, we need to solve $\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 10$.

Hence, the general solution is $C_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{5x} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{10x}$.

(b) The corresponding fundamental matrix solution is $\Phi = \begin{bmatrix} e^{5x} & 2e^{10x} \\ 3e^{5x} & e^{10x} \end{bmatrix}$.

(c) Note that $\Phi(0) = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1} = \frac{1}{5} \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} e^{5x} & 2e^{10x} \\ 3e^{5x} & e^{10x} \end{bmatrix} \frac{1}{5} \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -e^{5x} + 6e^{10x} & 2e^{5x} - 2e^{10x} \\ -3e^{5x} + 3e^{10x} & 6e^{5x} - e^{10x} \end{bmatrix}.$$

(d) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -e^{5x} + 6e^{10x} & 2e^{5x} - 2e^{10x} \\ -3e^{5x} + 3e^{10x} & 6e^{5x} - e^{10x} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -3e^{5x} + 8e^{10x} \\ -9e^{5x} + 4e^{10x} \end{bmatrix}$.