Final Exam

TUE, Nov 24

exam: 2^{00} – 3^{15} PM

upload work by 3^{45} PM
PDF

format
- as for midterm exams
- show-your-work problems ~ 3
- short answer problems ~ 6

practice
- review midterm exams
- practice problems + solutions
  (2 midterms + final)

tools
- calculators allowed — but: show work
- notes allowed — but: watch time

Questions?

Likely show-your-work problems:

1. compute matrix exponential
   + solve IVP
2. solve linear DE using
   power series method
3. heat equation (PDE)
   determine steady-state
   + transient solution

Problems 3 + 4
on HW 6

$$u(x,t) = v(x) + w(x,t)$$

"transient"
$\to 0$ as $t \to \infty$
(+ all derivatives)

non-constant coefficient

EG y_p $\qquad y'' + x^8 y = e^{6x}$ $\qquad \mid (D-6)$

root : 6

Find homogeneous DE for y_p : $(D-6)y = 0$

$y' = 6y$

"old" roots

"new" roots

$\frac{d}{dx}$

$$(D-6)\left[y'' + x^8 y\right] = 0$$

$$y''' + 8x^7 y + x^8 y' - 6y'' - 6x^8 y = 0$$

$$y''' \underbrace{-6}_{a(x)} y'' + \underbrace{x^8}_{b(x)} y' + \underbrace{(8x^7 - 6x^8)}_{c(x)} y = 0$$

EG $y'' = (5x+3)y' - 2y$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} \quad \xrightarrow{n \to n+1} \quad = \sum_{n=0}^{\infty} a_{n+1}(n+1) x^n$$

$$xy' = \sum_{n=1}^{\infty} a_n n x^n$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} \quad \xrightarrow{n \to n+2}$$

$$= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) x^n = 5\sum_{n=1}^{\infty} n a_n x^n + 3\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 2\sum_{n=0}^{\infty} a_n x^n$$

compare coefficients of x^n:

$$(n+2)(n+1)a_{n+2} = \overbrace{5n a_n}^{\text{if } n \geq 1} + 3(n+1)a_{n+1} - 2a_n$$

$n \to n-2$

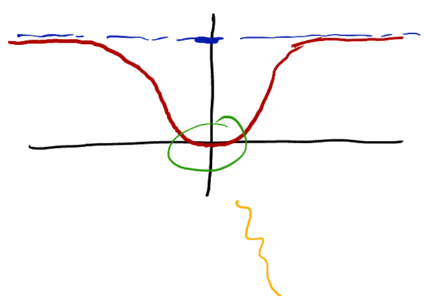
$$n(n-1)a_n = 5(n-2)a_{n-2} + 3(n-1)a_{n-1} - 2a_{n-2}$$

$$a_n = \boxed{\frac{3}{n}} a_{n-1} + \boxed{\frac{5(n-2)-2}{n(n-1)}} a_{n-2}$$

advanced

$$y = e^{-1/x^2}$$

not analytic at $x=0$



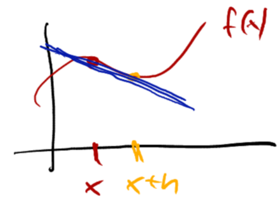
$$y \neq \sum_{n=0}^{\infty} a_n x^n$$

$$a_n = \frac{y^{(n)}(0)}{n!}$$

here: all $= 0$

discretization

$$Sf(x) = f(x+h)$$



$$f'(x) = \lim_{h\to 0} \boxed{\frac{f(x+h)-f(x)}{h}}$$

difference quotient

$$\boxed{\frac{1}{h}\left[S-1\right]f(x)}$$

$$f''(x)$$

$$\approx \left(\frac{1}{h}(S-1)\right)^2 f(x)$$

$$= \frac{1}{h^2}\left(S^2 - 2S + 1\right) f(x)$$

$$= \frac{1}{h^2}\left[f(x+2h) - 2f(x+h) + f(x)\right]$$

better: $\frac{1}{h^2}\left[f(x+h) - 2f(x) + f(x-h)\right] \approx f''(x)$

a_n
a_{n+1}
a_{n+2}
$\vdots$
$f(x)$
$f(x+h)$
$f(x+2h)$
$\vdots$

2D: $u_t = k\left(u_{xx} + u_{yy}\right)$

steady-state: $u_{xx} + u_{yy} = 0$ $\qquad u(x,y)$

$$\approx \frac{1}{h^2}\left[u(x+h,y) - 2u(x,y) + u(x-h,y)\right]$$

$$\approx \frac{1}{h^2}\left[\boxed{u(x+h,y)} + \boxed{u(x,y+h)} - 4\boxed{u(x,y)} + \boxed{u(x-h,y)} + u(x,y-h)\right]$$