Midterm #1

THU, Oct 1

exam: 2:00 – 3:15 PM

upload work by 3:45 PM

PDF

format

- check it out ☺ — link in email
- show-your-work problems — ~3
- short answer problems — ~6

practice

- review HW
- practice problems + solutions

tools

- calculators allowed — but: show work
- notes allowed — but: watch time

Questions?

- linear DEs with constant coefficients
  - homogeneous
  - inhomogeneous
- linear differential operators (normal form) $P(D)y = f(x)$, $xD \neq Dx$
- linear recurrence equations
  - Binet-like formulas
  - $\lim_{n\to\infty} a_{n+1}/a_n$
- systems of DEs / recurrences
  - M^n
  - $\exp(Mx)$

$a_n = a(n)$ $\quad \Phi_n \quad \Phi_0$

$y(x) \quad \Phi(x) \quad \Phi(0)$

EG $\quad D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

$$\exp(Dx) = \begin{bmatrix} e^x & 0 \\ 0 & e^{-2x} \end{bmatrix}$$

$\exp(Ax)$ fundamental matrix solution to

$y' = Ay \quad y(0) = I$

$\Phi(x)$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ -2y_2 \end{bmatrix}$

$y_1' = y_1 \quad y_1 = Ae^x$

$y_2' = -2y_2 \quad y_2 = Be^{-2x}$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} Ae^x \\ Be^{-2x} \end{bmatrix}$

$= A\begin{bmatrix} e^x \\ 0 \end{bmatrix} + B\begin{bmatrix} 0 \\ e^{-2x} \end{bmatrix}$

$\Psi(x) = \Phi(x)\,\Phi(0)^{-1}$

1-eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad e^x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

-2 eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad e^{-2x} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

λ-eigenvector v $\quad e^{\lambda x} v$

$\Phi(x)$: columns are solutions

$= \begin{bmatrix} e^x & 0 \\ 0 & e^{-2x} \end{bmatrix}$ $\quad \Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$e^{Ax} = \Phi(x)\,\Phi(0)^{-1}$$

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \frac{\Delta f(x)}{\Delta x}$$

$$\underset{m \times n}{\frac{d}{dx} A(x)} = \lim_{h \to 0} \frac{\overset{m \times n}{A(x+h)} - \overset{m \times n}{A(x)}}{h}$$

h: scalar

EG $y''' + 3y'' - 4y = e^{-2x} + 13x^2 \cdot e^{0x}$

$q(D)$ kills RHS:

homogeneous: $p(D)\,y = 0$

$\Rightarrow q(D)\,p(D)\,y = 0$

$p(D) = D^3 + 3D^2 - 4$

$= (D+2)^2(D-1)$

old roots: $-2, -2, 1$

new roots: $-2, 0, 0, 0$

$0, 0, 0$ ↔ $e^{0x} = 1$, x, x^2

$\Rightarrow y_p = Ax^2e^{-2x} + B + Cx + Dx^2$

find A, B, C, D by plugging into original DE

EG recurrence ...

$a_n = C_1 \lambda_1^n + C_2 \lambda_2^n$

determine from initial conds

$a_n \approx C_2 \lambda_2^n$ if $|\lambda_2| > |\lambda_1|$

$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lambda_2$ or whatever is the larger λ (in absolute value)

$\frac{a_{n+1}}{a_n} = \frac{C_2 \lambda_2^{n+1}}{C_2 \lambda_2^n} = \lambda_2$