Example 63. If $M = \begin{bmatrix} 3 & & & \\ & -2 & & \\ & & 5 & \\ & & & 1 \end{bmatrix}$, what is M^n?

Comment. Entries that are not printed are meant to be zero (to make the structure of the 4×4 matrix more visibly transparent).

Solution. $M^n = \begin{bmatrix} 3^n & & & \\ & (-2)^n & & \\ & & 5^n & \\ & & & 1 \end{bmatrix}$

If this isn't clear to you, multiply out M^2. What happens?

Preview: The corresponding system of differential equations

Review. Check out Examples 61 and 62 again.

Example 64. Write the (second-order) initial value problem $y'' = y' + 2y$, $y(0) = 0$, $y'(0) = 1$ as a first-order system.

Solution. If $\boldsymbol{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$, then $\boldsymbol{y}' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ y' + 2y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

This is exactly how we proceeded in Example 61.

Homework. Solve this IVP to find $y(x) = \frac{1}{3}(e^{2x} - e^{-x})$. Then compare with the next example.

Example 65. Let $M = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Solve $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution. In Example 62, we only need to replace 2^n by e^{2x} (root 2) and $(-1)^n$ by e^{-x} (root -1)!

(a) The general solution is $C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2x} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-x}$.

(b) A fundamental matrix solution is $\Phi(x) = \begin{bmatrix} e^{2x} & -e^{-x} \\ 2 \cdot e^{2x} & e^{-x} \end{bmatrix}$.

(c) $\boldsymbol{y}(x) = \frac{1}{3} \begin{bmatrix} e^{2x} - e^{-x} \\ 2 \cdot e^{2x} + e^{-x} \end{bmatrix}$

Preview. The special fundamental matrix M^n will be replaced by e^{Mx}, the **matrix exponential**.

Armin Straub
straub@southalabama.edu

Example 66. (homework)

(a) Write the recurrence $a_{n+3} - 4a_{n+2} + a_{n+1} + 6a_n = 0$ as a system $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ of (first-order) recurrences.

(b) Determine a fundamental matrix solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(c) Compute M^n.

Solution.

(a) If $\boldsymbol{a}_n = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix}$, then the RE becomes $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$ with $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix}$.

(b) Because we started with a single (third-order) equation, we can avoid computing eigenvectors and eigenvalues (indeed, we will find these as a byproduct).

By factoring the characteristic equation $N^3 - 4N^2 + N + 6 = (N-3)(N-2)(N+1)$, we find that the characteristic roots are $3, 2, -1$ (these are also precisely the eigenvalues of M).

Hence, $a_n = C_1 \cdot 3^n + C_2 \cdot 2^n + C_3 \cdot (-1)^n$ is the general solution to the initial RE.

Correspondingly, a fundamental matrix solution of the system is $\Phi_n = \begin{bmatrix} 3^n & 2^n & (-1)^n \\ 3 \cdot 3^n & 2 \cdot 2^n & -(-1)^n \\ 9 \cdot 3^n & 4 \cdot 2^n & (-1)^n \end{bmatrix}$.

Note. This tells us that $\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$ is a 3-eigenvector, $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ a 2-eigenvector, and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ a -1-eigenvector of M.

(c) Since $\Phi_{n+1} = M\Phi_n$, we have $\Phi_n = M^n \Phi_0$ so that $M^n = \Phi_n \Phi_0^{-1}$. This allows us to compute that:

$$M^n = \frac{1}{12} \begin{bmatrix} -6 \cdot 3^n + 12 \cdot 2^n + 6(-1)^n & -3 \cdot 3^n + 8 \cdot 2^n - 5(-1)^n & 3 \cdot 3^n - 4 \cdot 2^n + (-1)^n \\ -18 \cdot 3^n + 24 \cdot 2^n - 6(-1)^n & \ldots & \ldots \\ -54 \cdot 3^n + 48 \cdot 2^n + 6(-1)^n & \ldots & \ldots \end{bmatrix}$$

Systems of differential equations

Example 67. (review) Write the (second-order) differential equation $y'' = 2y' + y$ as a system of (first-order) differential equations.

Solution. If $\boldsymbol{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$, then $\boldsymbol{y}' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ 2y' + y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{y}$. For short, $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{y}$.

Comment. Hence, we care about systems of differential equations, even if we work with just one function.

Example 68. Write the (third-order) differential equation $y''' = 3y'' - 2y' + y$ as a system of (first-order) differential equations.

Solution. If $\boldsymbol{y} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$, then $\boldsymbol{y}' = \begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \\ 3y'' - 2y' + y \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \boldsymbol{y}$.

We can solve the system $\boldsymbol{y}' = M\boldsymbol{y}$ exactly as we solved $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

The only difference is that we replace each λ^n (for characteristic root / eigenvalue λ) with $e^{\lambda x}$. In fact, as shown in the examples below, we can translate back and forth at any stage.

To solve $\boldsymbol{y}' = M\boldsymbol{y}$, determine the eigenvectors of M.

- Each λ-eigenvector $\boldsymbol{v}$ provides a solution: $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$
- If there are enough eigenvectors, these combine to the general solution.

(systems of DEs) The unique solution to $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{c}$ is $\boldsymbol{y}(x) = e^{Mx}\boldsymbol{c}$.

- Here, e^{Mx} is the fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = I$ (with I the identity matrix).
- If $\Phi(x)$ is any fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$, then $e^{Mx} = \Phi(x)\Phi(0)^{-1}$.
- To construct a fundamental matrix solution $\Phi(x)$, we compute eigenvectors:
  Given a λ-eigenvector $\boldsymbol{v}$, we have the corresponding solution $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$.
  If there are enough eigenvectors, we can collect these as columns to obtain $\Phi(x)$.

Note. We are defining the **matrix exponential** e^{Mx} as the solution to an IVP. This is equivalent to how one can define the ordinary exponential e^x as the solution to $y' = y$, $y(0) = 1$.
[In a little bit, we will also discuss how to think about the matrix exponential e^{Mx} using power series.]

Comment. If there are not enough eigenvectors, then we know what to do (at least in principle): instead of looking only for solutions of the type $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$, we also need to look for solutions of the type $\boldsymbol{y}(x) = (\boldsymbol{v}x + \boldsymbol{w})e^{\lambda x}$. Note that this can only happen if an eigenvalue is a repeated root of the characteristic polynomial.

Important. Compare this to our method of solving systems of REs and for computing matrix powers M^n. Note that the above conclusion about systems of DEs can be deduced along the same lines as what we did for REs:

- If $\Phi(x)$ is a fundamental matrix solution, then so is $\Psi(x) = \Phi(x)C$ for every constant matrix C. (Why?!)
  Therefore, $\Psi(x) = \Phi(x)\Phi(0)^{-1}$ is a fundamental matrix solution with $\Psi(0) = \Phi(0)\Phi(0)^{-1} = I$.
  But e^{Mx} is defined to be the unique such solution, so that $\Psi(x) = e^{Mx}$.
- Let us look for solutions of $\boldsymbol{y}' = M\boldsymbol{y}$ of the form $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$. Note that $\boldsymbol{y}' = \lambda\boldsymbol{v}e^{\lambda x} = \lambda\boldsymbol{y}$.
  Plugging into $\boldsymbol{y}' = M\boldsymbol{y}$, we find $\lambda\boldsymbol{y} = M\boldsymbol{y}$.
  In other words, $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$ is a solution if and only if $\boldsymbol{v}$ is a λ-eigenvector of M.

Observe how the next example proceeds along the same lines as Example 60.

Important. In fact, we can translate back and forth (without additional computations) by simply replacing 3^n and $(-2)^n$ by e^{3x} and e^{-2x}.

Example 69. (homework) Let $M = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution. (See Example 60 for more details on the analogous computations.)

(a) Recall that each λ-eigenvector $\boldsymbol{v}$ of M provides us with a solution: namely, $\boldsymbol{y}(x) = \boldsymbol{v}e^{\lambda x}$.

We computed earlier that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -2$.

Hence, the general solution is $C_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{3x} + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2x}$.

(b) The corresponding fundamental matrix solution is $\Phi(x) = \begin{bmatrix} 2 \cdot e^{3x} & e^{-2x} \\ e^{3x} & e^{-2x} \end{bmatrix}$.

[Note that our general solution is precisely $\Phi(x)\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.]

(c) Since $\Phi(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, we have $\Phi(0)^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} 2 \cdot e^{3x} & e^{-2x} \\ e^{3x} & e^{-2x} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot e^{3x} - e^{-2x} & -2 \cdot e^{3x} + 2e^{-2x} \\ e^{3x} - e^{-2x} & -e^{3x} + 2e^{-2x} \end{bmatrix}.$$

Check. Let us verify the formula for e^{Mx} in the simple case $x = 0$: $e^{M0} = \begin{bmatrix} 2-1 & -2+2 \\ 1-1 & -1+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \cdot e^{3x} + 2e^{-2x} \\ -e^{3x} + 2e^{-2x} \end{bmatrix}$ (the second column of e^{Mx}).

Sage. We can compute the matrix exponential in Sage as follows:

```
>>> M = matrix([[8,-10],[5,-7]])
>>> exp(M*x)
```

$$\begin{pmatrix} (2\, e^{(5\,x)} - 1)\, e^{(-2\,x)} & -2\,(e^{(5\,x)} - 1)\, e^{(-2\,x)} \\ (e^{(5\,x)} - 1)\, e^{(-2\,x)} & -(e^{(5\,x)} - 2)\, e^{(-2\,x)} \end{pmatrix}$$

Note that this indeed matches the result of our computation.

[By the way, the variable x is pre-defined as a symbolic variable in Sage. That's why, unlike for n in the computation of M^n, we did not need to use `x = var('x')` first.]