Example 141. As in the previous example, consider the function $f(t) = 4 - t^2$, defined for $t \in [0, 2]$.

(a) Let $F(t)$ be the Fourier series of $f(t)$ (meaning the 2-periodic extension of $f(t)$). Determine $F(2)$, $F\left(\frac{5}{2}\right)$ and $F\left(-\frac{1}{2}\right)$.

(b) Let $G(t)$ be the Fourier cosine series of $f(t)$. Determine $G(2)$, $G\left(\frac{5}{2}\right)$ and $G\left(-\frac{1}{2}\right)$.

(c) Let $H(t)$ be the Fourier sine series of $f(t)$. Determine $H(2)$, $H\left(\frac{5}{2}\right)$ and $H\left(-\frac{1}{2}\right)$.

Solution.

(a) Note that the extension of $f(t)$ has discontinuities at $\ldots, -2, 0, 2, 4, \ldots$ (see plot in previous example) and recall that the Fourier series takes average values at these discontinuities:

$F(2) = \frac{1}{2}(F(2^-) + F(2^+)) = \frac{1}{2}(0 + 4) = 2$

$F\left(\frac{5}{2}\right) = F\left(\frac{5}{2} - 2\right) = f\left(\frac{1}{2}\right) = \frac{15}{4}$

$F\left(-\frac{1}{2}\right) = F\left(-\frac{1}{2} + 2\right) = f\left(\frac{3}{2}\right) = \frac{7}{4}$

(b) $G(2) = f(2) = 0$ (see plot!)

[Note that $G(2^+) = G(2^+ - 4) = G(-2^+) = G(2^-)$ where we used that G is even in the last step; in fact, we can show like this that the Fourier cosine series of a continuous function is always continuous.]

$G\left(\frac{5}{2}\right) = G\left(\frac{5}{2} - 4\right) = G\left(-\frac{3}{2}\right) = f\left(\frac{3}{2}\right) = \frac{7}{4}$

$G\left(-\frac{1}{2}\right) \overset{\text{even}}{=} G\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = \frac{15}{4}$

(c) $H(2) = \frac{1}{2}(f(2^-) - f(2^-)) = 0$ (see plot!)

[Note that $H(2^+) = H(2^+ - 4) = H(-2^+) = -H(2^-)$ where we used that H is odd in the last step; in fact, we can show like this that the Fourier sine series of a continuous function is always 0 at the jumps.]

$H\left(\frac{5}{2}\right) = H\left(\frac{5}{2} - 4\right) = H\left(-\frac{3}{2}\right) = -f\left(\frac{3}{2}\right) = -\frac{7}{4}$

$H\left(-\frac{1}{2}\right) \overset{\text{odd}}{=} -H\left(\frac{1}{2}\right) = -f\left(\frac{1}{2}\right) = -\frac{15}{4}$

Differentiating and integrating Fourier series

Theorem 142. If $f(t)$ is **continuous** and $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)$, then* $f'(t) = \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} b_n \cos\left(\frac{n\pi t}{L}\right) - \frac{n\pi}{L} a_n \sin\left(\frac{n\pi t}{L}\right)\right)$ (i.e., we can differentiate termwise).

Technical detail*: f' needs to be, e.g., piecewise smooth (so that it has a Fourier series itself).

Caution! We cannot simply differentiate termwise if $f(t)$ is lacking continuity. See the next example.

Comment. On the other hand, we can integrate termwise (going from the Fourier series of $f' = g$ to the Fourier series of $f = \int g$ because the latter will be continuous). This is illustrated in the example after the next.

Example 143. (caution!) The function $g(t) = \sum_{n \text{ odd}} \frac{4}{\pi n} \sin(n\pi t)$ from Example 139 is not continuous. For all values, except the discontinuities, we have $g'(t) = 0$. On the other hand, differentiating the Fourier series termwise, results in $4\sum_{n \text{ odd}} \cos(n\pi t)$, which diverges for most values of t (that's easy to check for $t = 0$). This illustrates that we cannot apply Theorem 142 because $g(t)$ is lacking continuity.

[The issues we are facing here can be fixed by generalizing the notion of function to distributions. (Maybe you have heard of the Dirac delta "function".)]

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Example 144. Let $h(t)$ be the 2-periodic function with $h(t)=|t|$ for $t\in[-1,1]$. Compute the Fourier series of $h(t)$.

Solution. We could just use the integral formulas to compute a_n and b_n. Since $h(t)$ is even (plot it!), we will find that $b_n=0$. Computing a_n is left as an exercise.

Solution. Note that $h(t)=\begin{cases}-t & \text{for } t\in(-1,0)\\ +t & \text{for } t\in(0,1)\end{cases}$ is continuous and $h'(t)=g(t)$, with $g(t)$ as in Example 139. Hence, we can apply Theorem 142 to conclude

$$h'(t)=g(t)=\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\sin(n\pi t) \quad\Longrightarrow\quad h(t)=\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi n}\left(-\frac{1}{\pi n}\right)\cos(n\pi t)+C,$$

where $C=\frac{a_0}{2}=\frac{1}{2}\int_{-1}^{1}h(t)\mathrm{d}t=\frac{1}{2}$ is the constant of integration. Thus, $h(t)=\frac{1}{2}-\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{\pi^2n^2}\cos(n\pi t)$.

Remark. Note that $t=0$ in the last Fourier series, gives us $\frac{\pi^2}{8}=\frac{1}{1}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$. As an exercise, you can try to find from here the fact that $\sum_{n\geqslant 1}\frac{1}{n^2}=\frac{\pi^2}{6}$. Similarly, we can use Fourier series to find that $\sum_{n\geqslant 1}\frac{1}{n^4}=\frac{\pi^4}{90}$.
Just for fun. These are the values $\zeta(2)$ and $\zeta(4)$ of the Riemann zeta function $\zeta(s)$. No such evaluations are known for $\zeta(3),\zeta(5),\ldots$ and we don't even know (for sure) whether these are rational numbers. Nobody believes these to be rational numbers, but it was only in 1978 that Apéry proved that $\zeta(3)$ is not a rational number.