Review. The heat equation: $u_t = k u_{xx}$

Example 157. Note that $u(x,t) = ax + b$ solves the heat equation.

Example 158. To get a feeling, let us find some solutions to $u_t = u_{xx}$ (for starters, $k = 1$).

- $u(x,t) = ax + b$ is a solution.
- For instance, $u(x,t) = e^t e^x$ is a solution.
  [Not a very interesting one for modeling heat flow because it increases exponentially in time.]
- More interesting are $u(x,t) = e^{-t}\cos(x)$ and $u(x,t) = e^{-t}\sin(x)$.
- More generally, $e^{-n^2 t}\cos(nx)$ and $e^{-n^2 t}\sin(nx)$ are solutions.
- Can you find further solutions?

Important observation. This actually reveals a strategy for solving the PDE $u_t = u_{xx}$ with conditions such as:

$$\begin{array}{ll} u(0,t) = u(\pi,t) = 0 & \text{(BC)} \\ u(x,0) = f(x), \quad x \in (0,\pi) & \text{(IC)} \end{array}$$

Namely, the solutions $u_n(x,t) = e^{-n^2 t}\sin(nx)$ all satisfy (BC).
It remains to satisfy (IC). Note that $u_n(x,0) = \sin(nx)$. To find $u(x,t)$ such that $u(x,0) = f(x)$, we can write $f(x)$ as a Fourier sine series (i.e. extend $f(x)$ to a 2π-periodic odd function):

$$f(x) = \sum_{n \geqslant 1} b_n \sin(nx)$$

Then $u(x,t) = \sum_{n \geqslant 1} b_n u_n(x,t) = \sum_{n \geqslant 1} b_n e^{-n^2 t}\sin(nx)$ solves the PDE $u_t = u_{xx}$ with (BC) and (IC).

Example 159. Find the unique solution $u(x,t)$ to: $\begin{array}{ll} u_t = u_{xx} & \text{(PDE)} \\ u(0,t) = u(\pi,t) = 0 & \text{(BC)} \\ u(x,0) = \sin(2x) - 7\sin(3x), \quad x \in (0,\pi) & \text{(IC)} \end{array}$

Solution. As we just observed (see next example for what to do in general), the functions $u_n(x,t) = e^{-n^2 t}\sin(nx)$ satisfy (PDE) and (BC) for any integer $n = 1, 2, 3, \ldots$
Since $u_n(x,0) = \sin(nx)$, we have $u_2(x,0) - 7u_3(x,0) = \sin(2x) - 7\sin(3x)$ as needed for (IC).
Therefore, $\text{(PDE)}+\text{(BC)}+\text{(IC)}$ is solved by

$$u(x,t) = u_2(x,t) - 7u_3(x,t) = e^{-4t}\sin(2x) - 7e^{-9t}\sin(3x).$$

Comment. Why did we restrict the integer n to the case $n \geqslant 1$?
[$n = 0$ just gives the zero function, and negative values don't give anything new because $u_{-n}(x,t) = -u_n(x,t)$.]

In the next example, we show that this idea can always be used to solve the heat equation.

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Example 160. Find the unique solution $u(x,t)$ to:
$$\begin{aligned} &u_t = k u_{xx} && \text{(PDE)}\\ &u(0,t) = u(L,t) = 0 && \text{(BC)}\\ &u(x,0) = f(x), \quad x \in (0,L) && \text{(IC)}\end{aligned}$$

Solution.

- We will first look for simple solutions of (PDE)+(BC) (and then we plan to take a combination of such solutions that satisfies (IC) as well). Namely, we look for solutions $u(x,t) = X(x)T(t)$. This approach is called **separation of variables** and it is crucial for solving other PDEs as well.
- Plugging into (PDE), we get $X(x)T'(t) = kX''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)}$.

  Note that the two sides cannot depend on x (because the right-hand side doesn't) and they cannot depend on t (because the left-hand side doesn't). Hence, they have to be constant. Let's call this constant $-\lambda$. Then, $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{const} =: -\lambda$.

  We thus have $X'' + \lambda X = 0$ and $T' + \lambda k T = 0$.
- Consider (BC). Note that $u(0,t) = X(0)T(t) = 0$ implies $X(0) = 0$.

  [Because otherwise $T(t) = 0$ for all t, which would mean that $u(x,t)$ is the dull zero solution.]

  Likewise, $u(L,t) = X(L)T(t) = 0$ implies $X(L) = 0$.
- So X solves $X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$. We know that, up to multiples, the only nonzero solutions are the eigenfunctions $X(x) = \sin\left(\frac{\pi n}{L}x\right)$ corresponding to the eigenvalues $\lambda = \left(\frac{\pi n}{L}\right)^2$, $n = 1, 2, 3 \ldots$.
- On the other hand, T solves $T' + \lambda k T = 0$, and hence $T(t) = e^{-\lambda k t} = e^{-\left(\frac{\pi n}{L}\right)^2 k t}$.
- Taken together, we have the solutions $u_n(x,t) = e^{-\left(\frac{\pi n}{L}\right)^2 k t} \sin\left(\frac{\pi n}{L}x\right)$ solving (PDE)+(BC).
- We wish to combine these in such a way that (IC) holds as well.

  At $t = 0$, $u_n(x,0) = \sin\left(\frac{\pi n}{L}x\right)$. All of these are $2L$-periodic.

  Hence, we extend $f(x)$, which is only given on $(0,L)$, to an odd $2L$-periodic function (its Fourier sine series!). By making it odd, its Fourier series will only involve sine terms: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{L}x\right)$. Note that

  $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x,$$

  where the first integral makes reference to the extension of $f(x)$ while the second integral only uses $f(x)$ on its original interval of definition.

Consequently, (PDE)+(BC)+(IC) is solved by

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{\pi n}{L}\right)^2 k t} \sin\left(\frac{\pi n}{L}x\right),$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

Example 161. Find the unique solution $u(x,t)$ to:

$$\begin{aligned} &u_t = 3u_{xx} && \text{(PDE)} \\ &u(0,t) = u(4,t) = 0 && \text{(BC)} \\ &u(x,0) = 5\sin(\pi x) - \sin(3\pi x), \quad x \in (0,4) && \text{(IC)} \end{aligned}$$

Solution. This is the case $k = 3$, $L = 4$ that we solved in Example 160 where we found that the functions

$$u_n(x,t) = e^{-\left(\frac{\pi n}{L}\right)^2 kt} \sin\left(\frac{\pi n}{L} x\right) = e^{-3\left(\frac{\pi n}{4}\right)^2 t} \sin\left(\frac{\pi n}{4} x\right)$$

solve (PDE)+(BC). Since $u_n(x,0) = \sin\left(\frac{\pi n}{4} x\right)$, we have

$$5u_4(x,0) - u_{12}(x,0) = 5\sin(\pi x) - \sin(3\pi x),$$

which is what we need for the right-hand side of (IC). Therefore, (PDE)+(BC)+(IC) is solved by

$$u(x,t) = 5u_4(x,t) - u_{12}(x,t) = 5e^{-3\pi^2 t}\sin(\pi x) - e^{-27\pi^2 t}\sin(3\pi x).$$