The inhomogeneous heat equation

We next indicate that we can similarly solve the inhomogeneous heat equation (with inhomogeneous boundary conditions).

Comment. We indicated earlier that

$$\begin{aligned} &u_t = k u_{xx} && \text{(PDE)} \\ &u(0,t) = a, \quad u(L,t) = b && \text{(BC)} \\ &u(x,0) = f(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

can be solved by realizing that $Ax + B$ solves (PDE).

Indeed, let $v(x) = a + \frac{b-a}{L}x$ (so that $v(0) = a$ and $v(L) = b$). We then look for a solution of the form $u(x,t) = v(x) + w(x,t)$. Note that $u(x,t)$ solves (PDE)+(BC)+(IC) if and only if $w(x,t)$ solves:

$$\begin{aligned} &w_t = k w_{xx} && \text{(PDE)} \\ &w(0,t) = 0, \quad w(L,t) = 0 && \text{(BC*)} \\ &w(x,0) = f(x) - v(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

This is the (homogeneous) heat equation that we know how to solve.

$v(x)$ is called the **steady-state solution** (it does not depend on time!) and $w(x,t)$ the **transient solution** (note that $w(x,t)$ and its partial derivatives tend to zero as $t \to \infty$ because of the boundary conditions (BC*)).

Example 165. Consider the heat flow problem:

$$\begin{aligned} &u_t = 3u_{xx} + 4x^2 && \text{(PDE)} \\ &u(0,t) = 1, \quad u_x(3,t) = -5 && \text{(BC)} \\ &u(x,0) = f(x), \quad x \in (0,3) && \text{(IC)} \end{aligned}$$

Determine the steady-state solution and spell out equations characterizing the transient solution.

Solution. We look for a solution of the form $u(x,t) = v(x) + w(x,t)$, where $v(x)$ is the steady-state solution and where $w(x,t)$ is the transient solution which (together with its derivatives) tends to zero as $t \to \infty$.

- Plugging into (PDE), we get $w_t = 3v'' + 3w_{xx} + 4x^2$. Letting $t \to \infty$, this becomes $0 = 3v'' + 4x^2$. Note that this also implies that $w_t = 3w_{xx}$.
- Plugging into (BC), we get $v(0) + w(0,t) = 1$ and $v'(3) + w_x(3,t) = -5$. Letting $t \to \infty$, these become $v(0) = 1$ and $v'(3) = -5$.
- Solving the ODE $0 = 3v'' + 4x^2$, we find

  $$v(x) = \iint -\frac{4}{3}x^2 \mathrm{d}x\mathrm{d}x = \int \left(-\frac{4}{9}x^3 + C\right)\mathrm{d}x = -\frac{1}{9}x^4 + Cx + D.$$

  The boundary conditions $v(0) = 1$ and $v'(3) = -5$ imply $D = 1$ and $-\frac{4}{9} \cdot 3^3 + C = -5$ (so that $C = 7$). In conclusion, the steady-state solution is $v(x) = -\frac{1}{9}x^4 + 1 + 7x$.

On the other hand, the transient solution $w(x,t)$ is characterized as the unique solution to:

$$\begin{aligned} &w_t = 3w_{xx} && \text{(PDE*)} \\ &w(0,t) = 0, \quad w_x(3,t) = 0 && \text{(BC*)} \\ &w(x,0) = f(x) - v(x) && \text{(IC*)} \end{aligned}$$

This homogeneous heat flow problem can now be solved using separation of variables.

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Example 166. For $t \geqslant 0$ and $x \in [0, 4]$, consider the heat flow problem:
$$\begin{aligned} u_t &= 2u_{xx} + e^{-x/2} \\ u_x(0,t) &= 3 \\ u(4,t) &= -2 \\ u(x,0) &= f(x) \end{aligned}$$

Determine the steady-state solution and spell out equations characterizing the transient solution.

Solution. We look for a solution of the form $u(x,t) = v(x) + w(x,t)$, where $v(x)$ is the steady-state solution and where the transient solution $w(x,t)$ tends to zero as $t \to \infty$ (as do its derivatives).

- Plugging into (PDE), we get $w_t = 2v'' + 2w_{xx} + e^{-x/2}$. Letting $t \to \infty$, this becomes $0 = 2v'' + e^{-x/2}$.
- Plugging into (BC), we get $w_x(0,t) + v'(0) = 3$ and $w(4,t) + v(4) = -2$.
  Letting $t \to \infty$, these become $v'(0) = 3$ and $v(4) = -2$.
- Solving the ODE $0 = 2v'' + e^{-x/2}$, we find
  $$v(x) = \iint -\frac{1}{2}e^{-x/2}\mathrm{d}x\mathrm{d}x = \int (e^{-x/2} + C)\mathrm{d}x = -2e^{-x/2} + Cx + D.$$
  The boundary conditions $v'(0) = 3$ and $v(4) = -2$ imply $C = 2$ and $-2e^{-2} + 8 + D = -2$.
  In conclusion, the steady-state solution is $v(x) = -2e^{-x/2} + 2x - 10 + 2e^{-2}$.

On the other hand, the transient solution $w(x,t)$ is characterized as the unique solution to:

$$\begin{aligned} &w_t = 2w_{xx} \\ &w_x(0,t) = 0, \quad w(4,t) = 0 \\ &w(x,0) = f(x) - v(x) \end{aligned}$$

Note. We know how to solve this homogeneous heat equation using separation of variables.

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Steady-state temperature: The Laplace equation

(2D and 3D heat equation) In higher dimensions, the heat equation takes the form $u_t = k(u_{xx} + u_{yy})$ or $u_t = k(u_{xx} + u_{yy} + u_{zz})$.

The heat equation is often written as $u_t = k\Delta u$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ (2D) or $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (3D) is the **Laplace operator** you may know from Calculus III.

Other notations. $\Delta u = \operatorname{div} \operatorname{grad} u = \nabla \cdot \nabla u = \nabla^2 u$

If temperature is steady, then $u_t = 0$. Hence, the steady-state temperature $u(x, y)$ must satisfy the PDE $u_{xx} + u_{yy} = 0$.

(Laplace equation, 2D)

$$u_{xx} + u_{yy} = 0$$

Comment. The Laplace equation is so important that its solutions have their own name: **harmonic functions**.

It is also known as the "potential equation"; satisfied by electric/gravitational potential functions. (More generally, such potentials, if not in the vacuum, satisfy the **Poisson equation** $u_{xx} + u_{yy} = f(x, y)$, the inhomogeneous version of the Laplace equation.)

Recall from Calculus III (if you have taken that class) that the gradient of a scalar function $f(x, y)$ is the vector field $\boldsymbol{F} = \operatorname{grad} f = \nabla f = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}$. One says that $\boldsymbol{F}$ is a **gradient field** and f is a **potential function** for $\boldsymbol{F}$ (for instance, $\boldsymbol{F}$ could be a gravitational field with gravitational potential f).

The divergence of a vector field $\boldsymbol{G} = \begin{bmatrix} g(x, y) \\ h(x, y) \end{bmatrix}$ is $\operatorname{div} \boldsymbol{G} = g_x + h_y$. One also writes $\operatorname{div} \boldsymbol{G} = \nabla \cdot \boldsymbol{G}$.

The gradient field of a scalar function f is divergence-free if and only if f satisfies the Laplace equation $\Delta f = 0$.

One way to describe a unique solution to the Laplace equation within a region is by specifying the values of $u(x, y)$ along the boundary of that region.

This is particularly natural for steady-state temperatures profiles of a region R. The Laplace equation governs how temperature behaves inside the region but we need to also prescribe the temperature on the boundary.

The PDE with such a boundary condition is called a Dirichlet problem:

(Dirichlet problem)

$u_{xx} + u_{yy} = 0$ within region R

$u(x, y) = f(x, y)$ on boundary of R

In general. A Dirichlet problem consists of a PDE, that needs to hold within a region R, and prescribed values on the boundary of that region ("Dirichlet boundary conditions").

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