Midterm #1 – Practice

MATH 332 — Differential Equations II
Midterm: Wednesday, Oct 2, 2024

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any mathematical typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Reminder. No notes, calculators (with the exception of one that can only do basic arithmetic—no graphing or additional algebraic capabilities) or tools of any kind will be permitted on the midterm exam.

Problem 1. Let $M = \begin{bmatrix} 1 & 4 \\ 6 & -1 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(b) Determine a fundamental matrix solution to $\boldsymbol{a}_{n+1} = M\boldsymbol{a}_n$.

(c) Compute M^n.

(d) Without further computations, determine e^{Mt}.

(e) Determine all equilibrium points of $\begin{bmatrix} x \\ y \end{bmatrix}' = M\begin{bmatrix} x \\ y \end{bmatrix}$ and their stability.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 4 \\ 6 & -1-\lambda \end{bmatrix}\right) = (1-\lambda)(-1-\lambda) - 24 = \lambda^2 - 25 = (\lambda - 5)(\lambda + 5)$

Hence, the eigenvalues are $\lambda = 5$ and $\lambda = -5$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = 5$, we need to solve $\begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 5$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = -5$, we need to solve $\begin{bmatrix} 6 & 4 \\ 6 & 4 \end{bmatrix}\boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ is an eigenvector for $\lambda = -5$.

Hence, the general solution is $C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}5^n + C_2\begin{bmatrix} -2 \\ 3 \end{bmatrix}(-5)^n$.

(b) The corresponding fundamental matrix solution is $\Phi_n = \begin{bmatrix} 5^n & -2(-5)^n \\ 5^n & 3(-5)^n \end{bmatrix}$.

(c) Note that $\Phi_0 = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$, so that $\Phi_0^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$. It follows that

$$M^n = \Phi_n \Phi_0^{-1} = \begin{bmatrix} 5^n & -2(-5)^n \\ 5^n & 3(-5)^n \end{bmatrix}\frac{1}{5}\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \cdot 5^n + 2(-5)^n & 2 \cdot 5^n - 2(-5)^n \\ 3 \cdot 5^n - 3(-5)^n & 2 \cdot 5^n + 3(-5)^n \end{bmatrix}.$$

(d) We just need to replace 5^n and $(-5)^n$ by e^{5t} and e^{-5t} respectively, to obtain

$$e^{Mt} = \frac{1}{5}\begin{bmatrix} 3e^{5t} + 2e^{-5t} & 2e^{5t} - 2e^{-5t} \\ 3e^{5t} - 3e^{-5t} & 2e^{5t} + 3e^{-5t} \end{bmatrix}.$$

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(e) The only equilibrium point is $(0,0)$ and it is unstable.

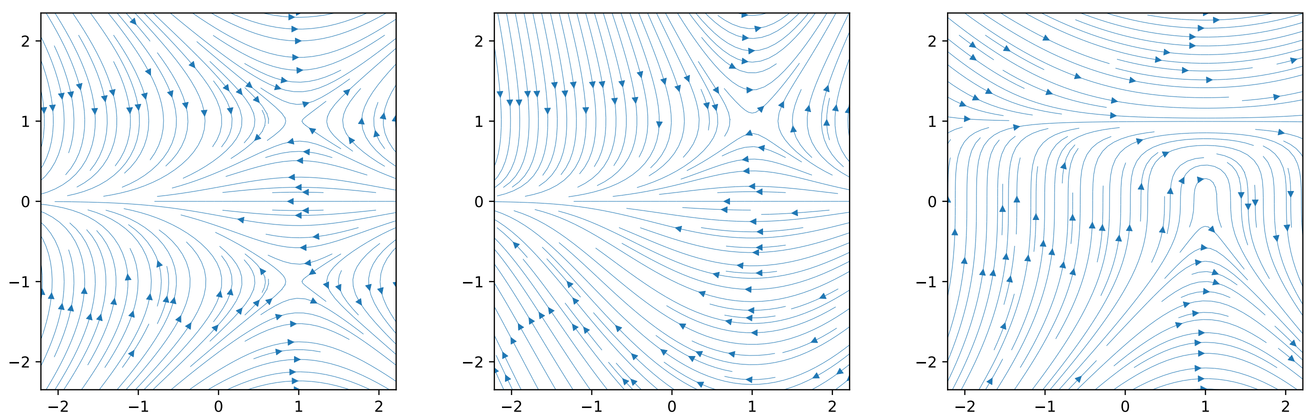
Since M is invertible, solving $M\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}$ we only get the unique solution $\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}$, which means that only $(0,0)$ is an equilibrium point. On the other hand, looking at e^{Mt} we see that the eigenvalues of M are 5 and -5. Because one eigenvalue is positive, the equilibrium point is unstable.

(For instance, we see that $C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{5t}$ is a solution for any C_1 which means that a trajectory in the phase portrait is the line through the origin spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; however, this trajectory "flows away" from $(0,0)$ because $e^{5t} \to \infty$ as $t \to \infty$. In fact, the general solution is $C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{5t} + C_2\begin{bmatrix} -2 \\ 3 \end{bmatrix}e^{-5t}$ and we see that all corresponding trajectories "flow away" from the origin unless $C_1 = 0$. Since there are trajectories (the ones corresponding to $C_1 = 0$) that do "flow into" the origin, the origin is a *saddle point*. Saddle points are always unstable.)

Problem 2.

(a) Circle the phase portrait below which belongs to $\frac{dx}{dt} = y^2 - 1$, $\frac{dy}{dt} = y \cdot (x-1)$.

(b) Determine all equilibrium points and classify the stability of each.



Solution.

(a) The first plot is the correct one.

An easy way to tell in this case is to compute the equilibrium points first.

Another way to tell would be to look at certain points that distinguish the plots:

- For instance, looking at the point $(1,-2)$, the equations tell us that $\frac{dx}{dt} = y^2 - 1 = 3$, $\frac{dy}{dt} = y \cdot (x-1) = 0$ (hence the slope is $\frac{dy}{dx} = \frac{0}{3} = 0$ but that doesn't help in telling the plots apart). That means the trajectory through $(1,-2)$ is moving in direction $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ which is horizontal with the arrow pointing to the right (positive x-direction). This means that the second plot cannot be the correct one.

- As another example, we can look at the point $(-2,1)$. The equations tell us that $\frac{dx}{dt} = y^2 - 1 = 0$, $\frac{dy}{dt} = y \cdot (x-1) = -3$. That means the trajectory through $(-2,1)$ is moving in direction $\begin{bmatrix} 0 \\ -3 \end{bmatrix}$ which is vertical with the arrow pointing down (negative y-direction). This means that the third plot cannot be the correct one.

(b) We solve $y^2 - 1 = 0$ (that is, $y = \pm 1$) and $y(x-1) = 0$ (that is, $x = 1$ or $y = 0$).

The only possibilities are $x = 1$ and $y = \pm 1$.

We conclude that the equilibrium points are $(1,1)$ and $(1,-1)$.

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Both equilibrium points are unstable (because some nearby solutions “flow away” from each). (More precisely, both are an example of a saddle.)

Problem 3.

(a) Write the differential equation $y''' + 7y'' - 3y' + y = 0$ as a system of (first-order) differential equations.

(b) Consider the following system of initial value problems:

$$\begin{aligned} y_1'' &= 3y_1' + 2y_2' - 5y_1 \\ y_2'' &= y_1' - y_2' + 3y_2 \end{aligned} \qquad y_1(0) = 1,\ y_1'(0) = -2,\ y_2(0) = 3,\ y_2'(0) = 0$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y},\ \boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution.

(a) Write $y_1 = y,\ y_2 = y'$ and $y_3 = y''$.

Then, $y''' + 7y'' - 3y' + y = 0$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -y_1 + 3y_2 - 7y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & -7 \end{bmatrix} \boldsymbol{y}$.

(b) Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 0 & 3 & 2 \\ 0 & 3 & 1 & -1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \end{bmatrix}.$$

Problem 4. Let $M = \begin{bmatrix} 11 & -2 \\ 3 & 4 \end{bmatrix}$.

(a) Determine the general solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(b) Determine a fundamental matrix solution to $\boldsymbol{y}' = M\boldsymbol{y}$.

(c) Compute e^{Mx}.

(d) Solve the initial value problem $\boldsymbol{y}' = M\boldsymbol{y}$ with $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(e) Determine all equilibrium points of $\boldsymbol{y}' = M\boldsymbol{y}$ and their stability.

Solution.

(a) We determine the eigenvectors of M. The characteristic polynomial is:

$\det(M - \lambda I) = \det\left(\begin{bmatrix} 11-\lambda & -2 \\ 3 & 4-\lambda \end{bmatrix}\right) = (11-\lambda)(4-\lambda) + 6 = \lambda^2 - 15\lambda + 50 = (\lambda - 5)(\lambda - 10)$

Hence, the eigenvalues are $\lambda = 5$ and $\lambda = 10$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = 5$, we need to solve $\begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0}$.

  Hence, $\boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector for $\lambda = 5$.

- To find an eigenvector $\boldsymbol{v}$ for $\lambda = 10$, we need to solve $\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0}$.

Hence, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 10$.

Hence, the general solution is $C_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{5x} + C_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{10x}$.

(b) The corresponding fundamental matrix solution is $\Phi = \begin{bmatrix} e^{5x} & 2e^{10x} \\ 3e^{5x} & e^{10x} \end{bmatrix}$.

(c) Note that $\Phi(0) = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, so that $\Phi(0)^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix}$. It follows that

$$e^{Mx} = \Phi(x)\Phi(0)^{-1} = \begin{bmatrix} e^{5x} & 2e^{10x} \\ 3e^{5x} & e^{10x} \end{bmatrix}\frac{1}{5}\begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -e^{5x} + 6e^{10x} & 2e^{5x} - 2e^{10x} \\ -3e^{5x} + 3e^{10x} & 6e^{5x} - e^{10x} \end{bmatrix}.$$

(d) The solution to the IVP is $\boldsymbol{y}(x) = e^{Mx}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -e^{5x} + 6e^{10x} & 2e^{5x} - 2e^{10x} \\ -3e^{5x} + 3e^{10x} & 6e^{5x} - e^{10x} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3e^{5x} + 8e^{10x} \\ -9e^{5x} + 4e^{10x} \end{bmatrix}$.

(e) The only equilibrium point is $(0, 0)$ and it is unstable.

Since M is invertible, solving $M\boldsymbol{y} = \boldsymbol{0}$ we only get the unique solution $\boldsymbol{y} = \boldsymbol{0}$, which means that only $(0, 0)$ is an equilibrium point. On the other hand, the general solution shows that every solution gets larger in magnitude as $x \to \infty$ because both e^{5x} and e^{10x} approach ∞.

Problem 5.

(a) Find the general solution to $y^{(5)} - 4y^{(4)} + 5y''' - 2y'' = 0$.

(b) Find the general solution to $y''' - y = e^x + 7$.

(c) Solve $y'' + 2y' + y = 2e^{2x} + e^{-x}$, $y(0) = -1$, $y'(0) = 2$.

(d) Find the general solution to $y'' - 4y' + 4y = 3e^{2x}$.

(e) Consider a homogeneous linear differential equation with constant real coefficients which has order 6. Suppose $y(x) = x^2e^{2x}\cos(x)$ is a solution. Write down the general solution.

(f) Write down a homogeneous linear differential equation satisfied by $y(x) = 1 - 5x^2e^{-2x}$.

(g) Let y_p be any solution to the inhomogeneous linear differential equation $y'' + xy = e^x$. Find a homogeneous linear differential equation which y_p solves. *Hint:* Do not attempt to solve the DE.

Solution.

(a) The characteristic polynomial $p(D) = D^5 - 4D^4 + 5D^3 - 2D^2 = D^2(D-1)^2(D-2)$ has roots $0, 0, 1, 1, 2$.

Hence, the general solution is $y(x) = c_1 + c_2x + (c_3 + c_4x)e^x + c_5e^{2x}$.

(b) The characteristic polynomial $p(D) = D^3 - 1$ of the associated homogeneous DE has roots 1 and $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. These are the "old" roots.

The "new" roots coming from $e^x + 7$ are $0, 1$. Hence, there has to be a particular solution of the form $y_p = Axe^x + B$. To find the values of A, B, we plug into the DE.

$y_p' = A(x+1)e^x$, $y_p'' = A(x+2)e^x$, $y_p''' = A(x+3)e^x$

$y_p''' - y_p = 3Ae^x - B \overset{!}{=} e^x + 7$

Consequently, $A = \frac{1}{3}$, $B = -7$.

Hence, the general solution is $y(x) = -7 + (c_1 + \frac{1}{3}x)e^x + c_2e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$.

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(c) The characteristic polynomial $p(D) = D^2 + 2D + 1$ of the associated homogeneous DE has roots $-1, -1$. These are the "old" roots.

The "new" roots coming from $2e^{2x} + e^{-x}$ are $-1, 2$. Hence, there has to be a particular solution of the form $y_p = Ae^{2x} + Bx^2e^{-x}$. To find the values of A, B, we plug into the DE.

$y_p' = 2Ae^{2x} + B(2x - x^2)e^{-x}$, $y_p'' = 4Ae^{2x} + B(2 - 4x + x^2)e^{-x}$

$y_p'' + 2y_p' + y_p = 9Ae^{2x} + 2Be^{-x} \overset{!}{=} 2e^{2x} + e^{-x}$

Consequently, $A = \frac{2}{9}$, $B = \frac{1}{2}$.

Hence, the general solution is $y(x) = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x} + c_1e^{-x} + c_2xe^{-x}$. Now, we use the initial values to find the values for c_1 and c_2:

$y(0) = \frac{2}{9} + c_1 \overset{!}{=} -1$, so that $c_1 = -\frac{11}{9}$.

$y'(0) = \left[\frac{4}{9}e^{2x} + \left(x - \frac{1}{2}x^2\right)e^{-x} + \frac{11}{9}e^{-x} + c_2(1 - x)e^{-x}\right]_{x=0} = \frac{5}{3} + c_2 \overset{!}{=} 2$, so that $c_2 = \frac{1}{3}$.

In conclusion, the unique solution to the IVP is $y(x) = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x} - \frac{11}{9}e^{-x} + \frac{1}{3}xe^{-x}$.

(d) The characteristic polynomial $p(D) = D^2 - 4D - 4$ of the associated homogeneous DE has "old" roots $2, 2$.

The "new" roots coming from $3e^{2x}$ are 2. Hence, there has to be a particular solution of the form $y_p = Ax^2e^{2x}$. To find the value of A, we plug into the DE.

$y_p' = 2A(x + x^2)e^{2x}$, $\quad y_p'' = 2A(1 + 4x + 2x^2)e^{2x}$

$y_p'' - 4y_p' + 4y_p = [2A(1 + 4x + 2x^2) - 8A(x + x^2) + 4Ax^2]e^{2x} = 2Ae^{2x} \overset{!}{=} 3e^{2x}$. Consequently, $A = \frac{3}{2}$.

Hence, the general solution is $\left(c_1 + c_2x + \frac{3}{2}x^2\right)e^{2x}$.

(e) $y(x) = x^2e^{2x}\cos(x)$ is a solution of $p(D)y = 0$ if and only if $2 \pm i$ are three times repeated roots of the characteristic polynomial $p(D)$. Since the order of the DE is 6, there can be no further roots.

The general solution of this DE is $y(x) = (c_1 + c_2x + c_3x^2)e^{2x}\cos(x) + (c_4 + c_5x + c_6x^2)e^{2x}\sin(x)$.

(f) $y(x) = 1 - 5x^2e^{-2x}$ is a solution of $p(D)y = 0$ if and only if $-2, -2, -2, 0$ are roots of the characteristic polynomial $p(D)$. Hence, the simplest DE is obtained from $p(D) = D(D + 2)^3 = D^4 + 6D^3 + 12D^2 + 8D$.

The corresponding recurrence is $y^{(4)} + 6y''' + 12y'' + 8y' = 0$.

(g) To kill e^x, we apply $D - 1$ to both sides of the DE $y'' + xy = e^x$.

The result is the homogeneous linear DE $y''' - y'' + xy' + (1 - x)y = 0$.

Comment. If we are comfortable computing with operators, we can apply the relation $Dx = xD + 1$, to $(D - 1)(D^2 + x) = D^3 - D^2 + Dx - x = D^3 - D^2 + xD + 1 - x$ to reach the same conclusion.

Problem 6.

(a) Write down a (homogeneous linear) recurrence equation satisfied by $a_n = 3^n - 2^n$.

(b) Write down a (homogeneous linear) recurrence equation satisfied by $a_n = n^23^n - 2^n$.

Solution.

(a) $a_n = 3^n - 2^n$ is a solution of $p(N)a_n = 0$ if and only if both 3 and 2 are a root of the characteristic polynomial $p(N)$. Hence, the simplest recurrence is obtained from $p(N) = (N - 2)(N - 3) = N^2 - 5N + 6$.

The corresponding recurrence is $a_{n+2} = 5a_{n+1} - 6a_n$.

(b) $a_n = n^2 3^n - 2^n$ is a solution of $p(N) a_n = 0$ if and only if 3 (repeated three times) and 2 are a root of the characteristic polynomial $p(N)$. Hence, the simplest recurrence is obtained from $p(N) = (N-2)(N-3)^3$.

The corresponding recurrence is $(N-2)(N-3)^3 a_n = 0$.

[Spelled out, this is $a_{n+4} = 11a_{n+3} - 45a_{n+2} + 81a_{n+1} - 54a_n$.]

Problem 7. Consider the sequence a_n defined by $a_{n+2} = a_{n+1} + 6a_n$ and $a_0 = 3$, $a_1 = -1$.

(a) Determine the first few terms of the sequence.

(b) Find a Binet-like formula for a_n.

(c) Determine $\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$.

Solution.

(a) $a_2 = 17$, $a_3 = 11$

(b) The recursion can be written as $p(N) a_n = 0$ where $p(N) = N^2 - N - 6$ has roots $3, -2$.

Hence, $a_n = \alpha_1 3^n + \alpha_2 (-2)^n$ and we only need to figure out the two unknowns α_1, α_2. We can do that using the two initial conditions: $a_0 = \alpha_1 + \alpha_2 = 3$, $a_1 = 3\alpha_1 - 2\alpha_2 = -1$.

Solving, we find $\alpha_1 = 1$ and $\alpha_2 = 2$ so that, in conclusion, $a_n = 3^n + 2 \cdot (-2)^n$.

(c) It follows from the Binet-like formula that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 3$.

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